The many meanings of mathematical concepts

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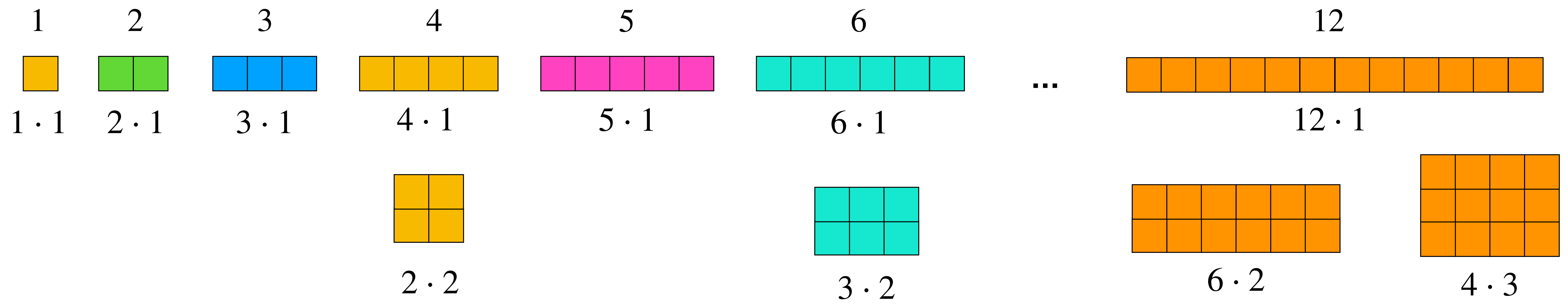
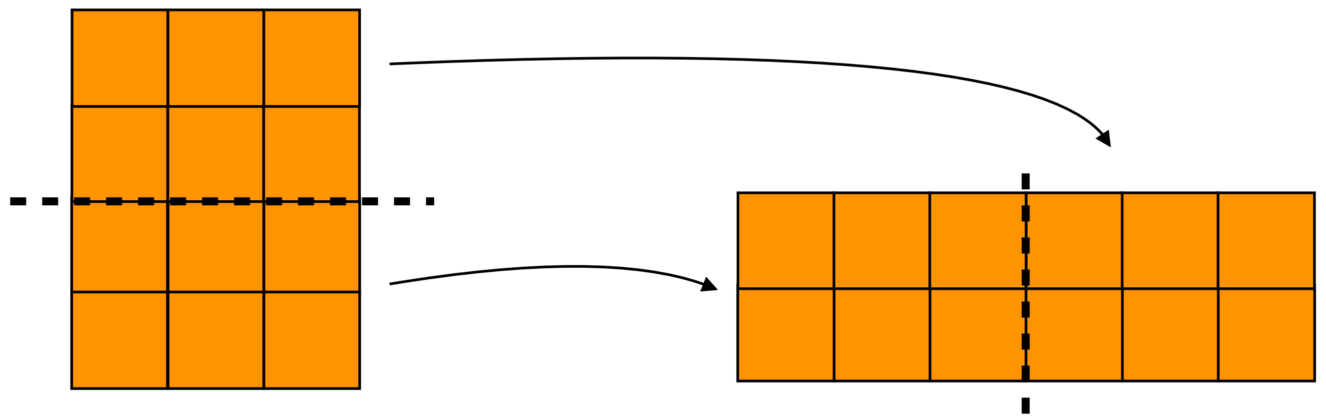
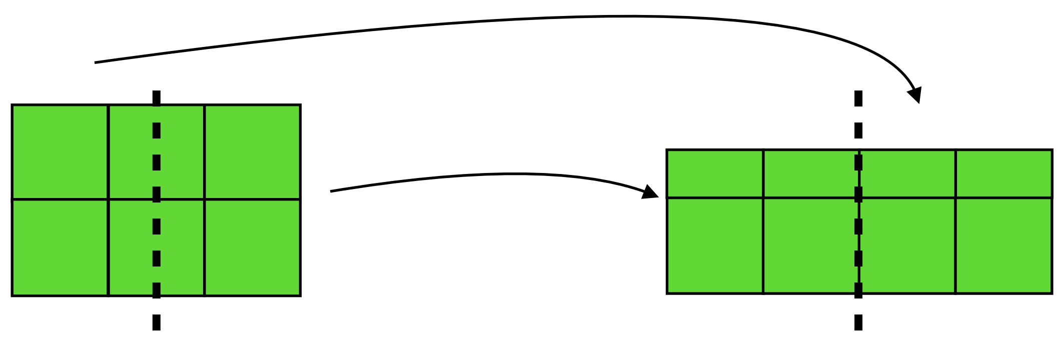
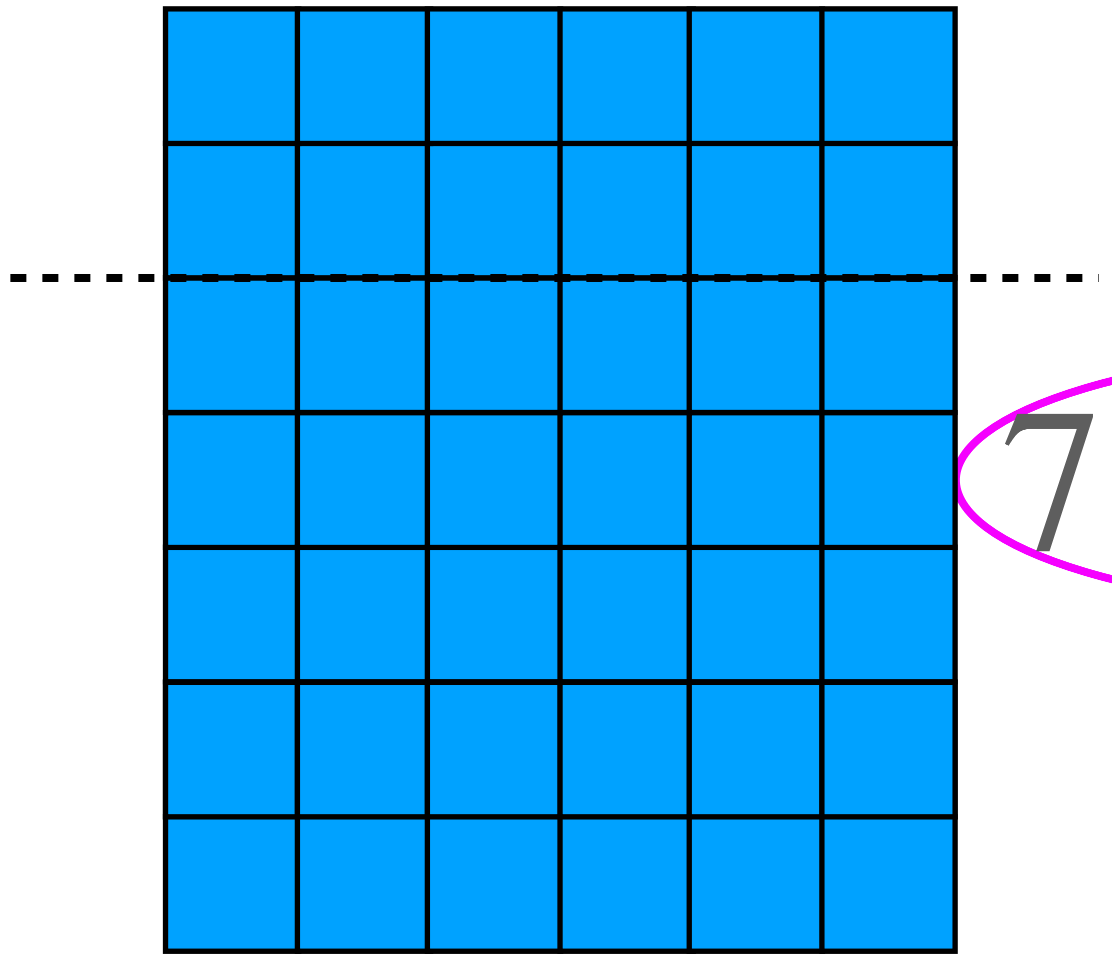




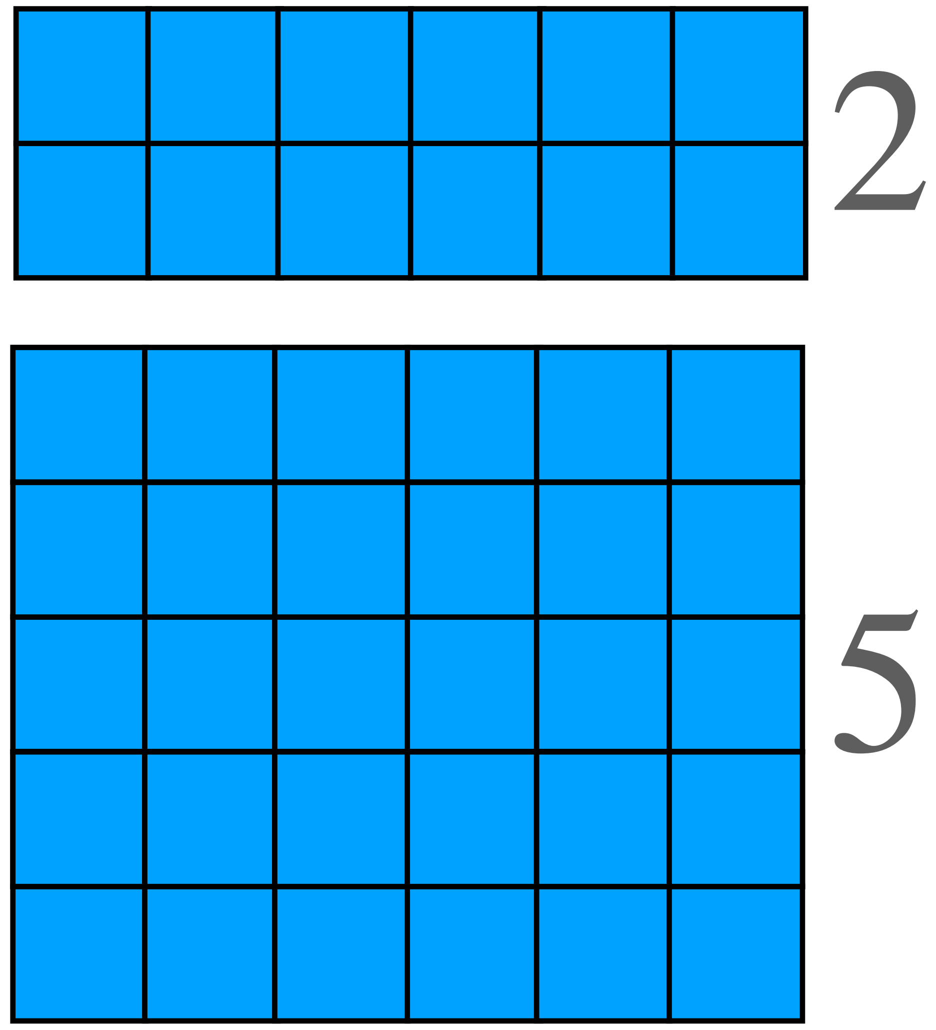
$$3 \cdot 4 = 6 \cdot 2$$



$$3 \cdot 2 = 4 \cdot 1\frac{1}{2}$$

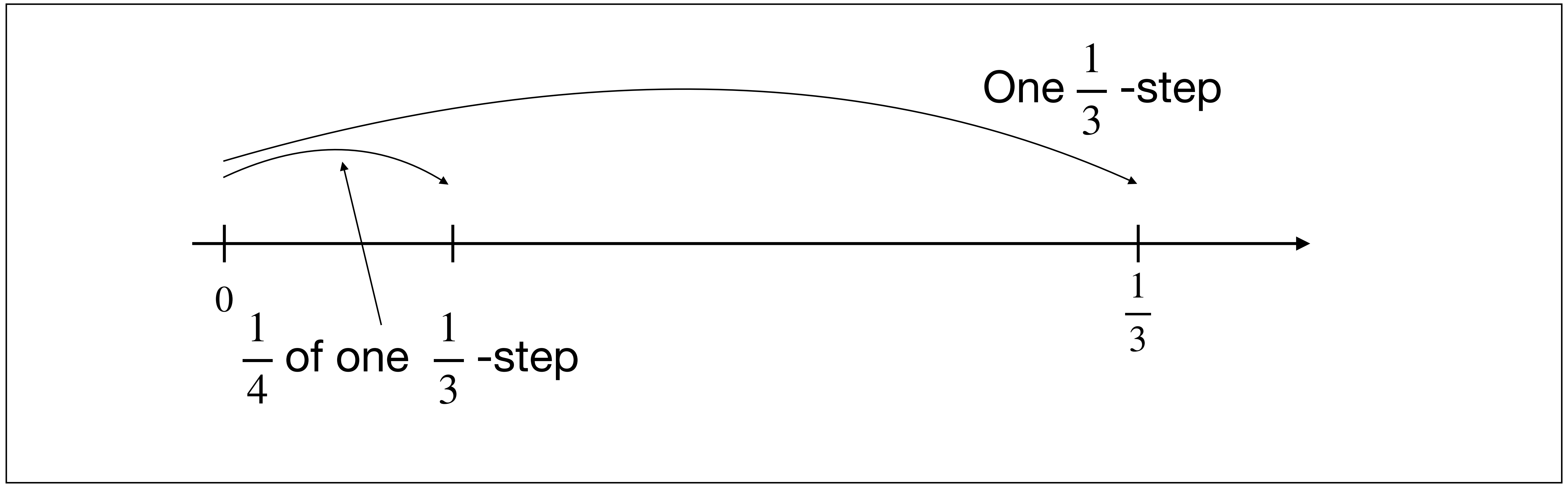


$$7 = 5 + 2$$

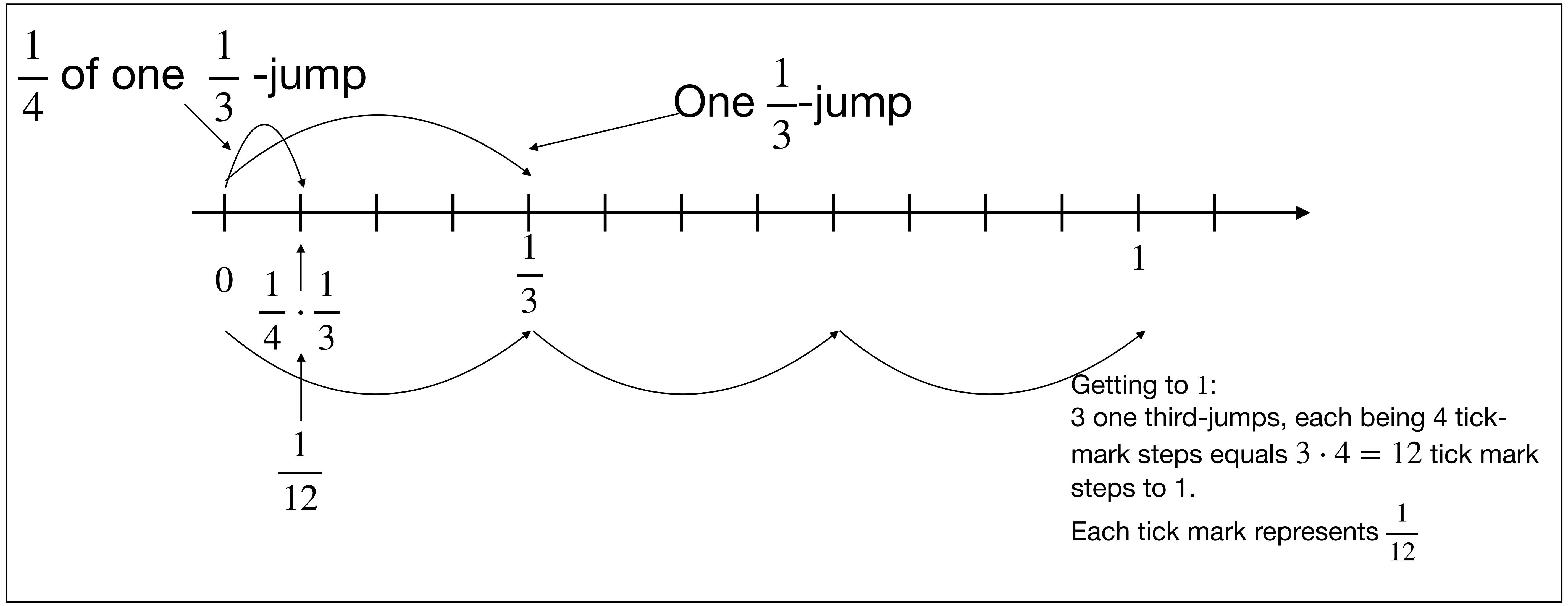


$$6 \cdot 7 = 6 \cdot 5 + 2 = 6 \cdot 5 + 6 \cdot 2$$

$$6 \cdot 7 = 6 \cdot (5 + 2) = 6 \cdot 5 + 6 \cdot 2$$



One $\frac{1}{3}$ -step
0
$\frac{1}{3}$
$\frac{1}{4}$ of one $\frac{1}{3}$ -step



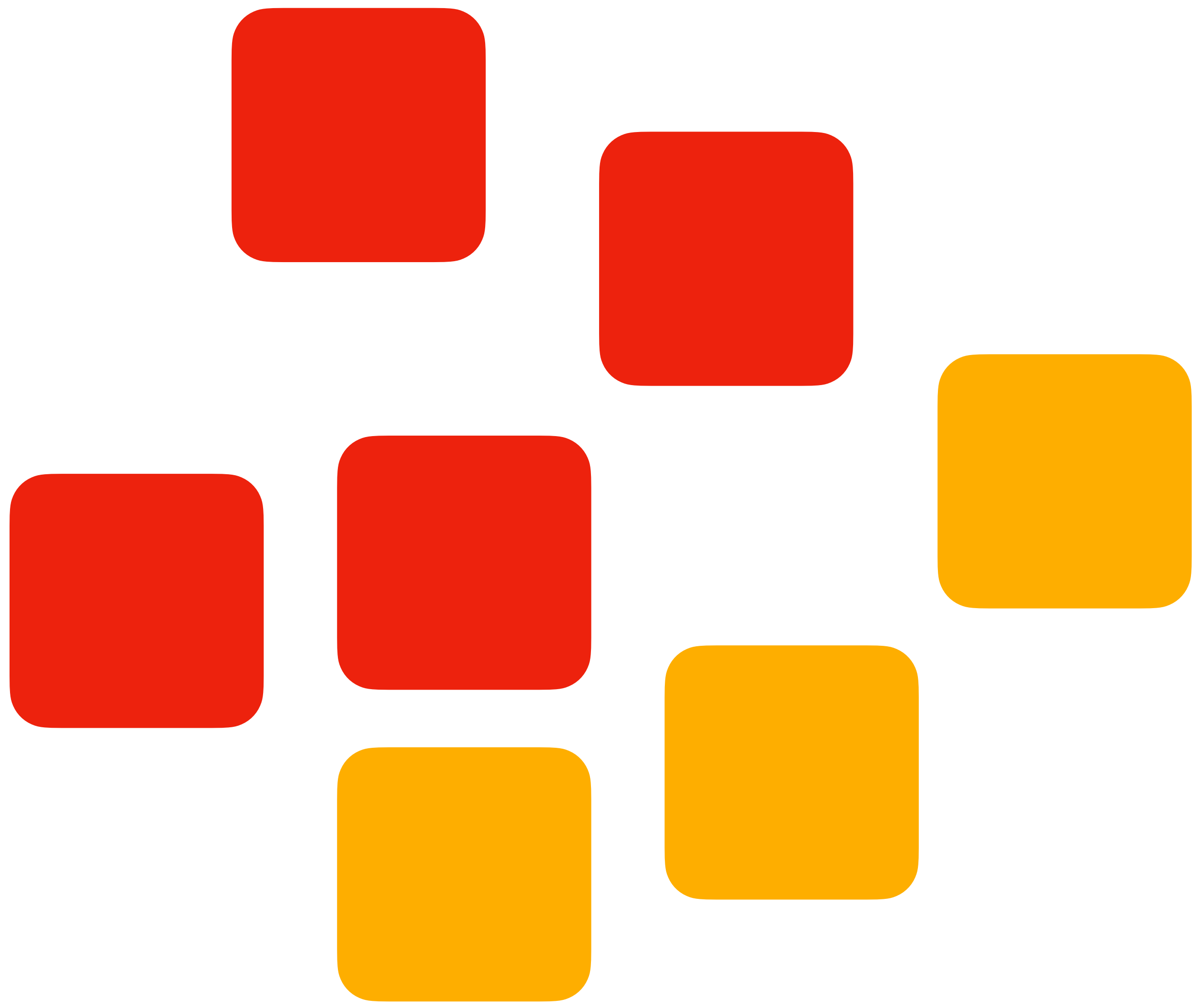
$\frac{1}{4}$ of one $\frac{1}{3}$ -jump
One $\frac{1}{3}$-jump
0
$\frac{1}{4} \cdot \frac{1}{3}$
$\frac{1}{3}$
1
$\frac{1}{12}$
Getting to 1:
3 one third-jumps, each being 4 tick-mark steps equals $3 \cdot 4 = 12$ tick mark steps to 1.
Each tick mark represents $\frac{1}{12}$

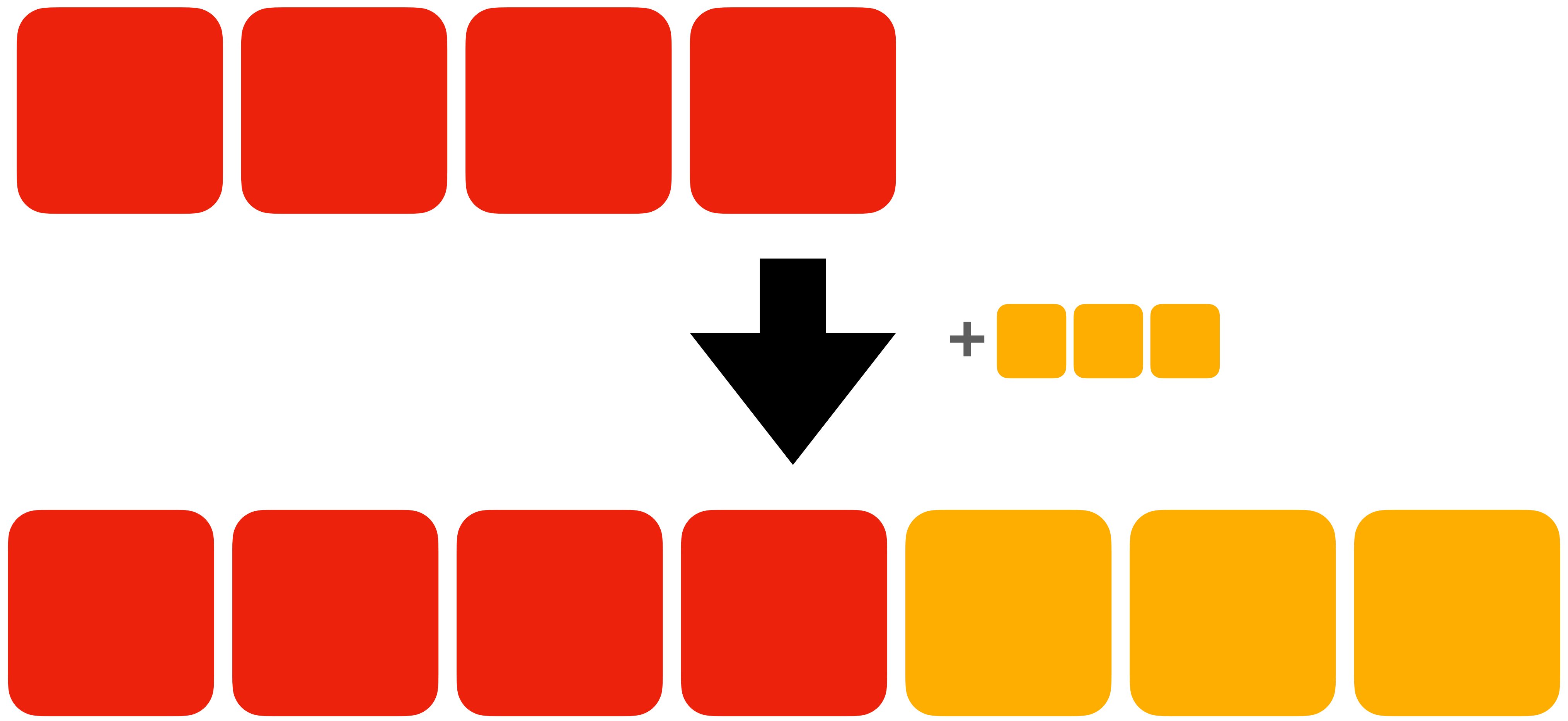
	Little Bear	**Big Bear**	**Mama Bear**
waterbottles	$\cdot 6$ $\cdot 2$ 1	$1 \cdot 2 = 2$ $\cdot 3$	$1 \cdot 6 = 6$
almonds	5	10	30
bisquits	2	4	12
apples	$\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2}$ $\cdot \frac{1}{2}$	1 /2	$3 = 6 \cdot \frac{1}{2} = 3 \cdot 1$
nuts	5	10	$30 = 6 \cdot 5 = 3 \cdot 10$
grapes	3 $\cdot \frac{1}{2}$	$\cdot \frac{1}{6}$ $6 = 2 \cdot 3 = \frac{1}{3} \cdot 18$	$\cdot \frac{1}{3}$ 18
buns	$\frac{1}{2}$	$1 = 2 \cdot \frac{1}{2} = \frac{1}{3} \cdot 3$	3

$\cdot 2$

Situations and iconic schematic imagery

Addition - ”lägga ihop” / ”lägga till”



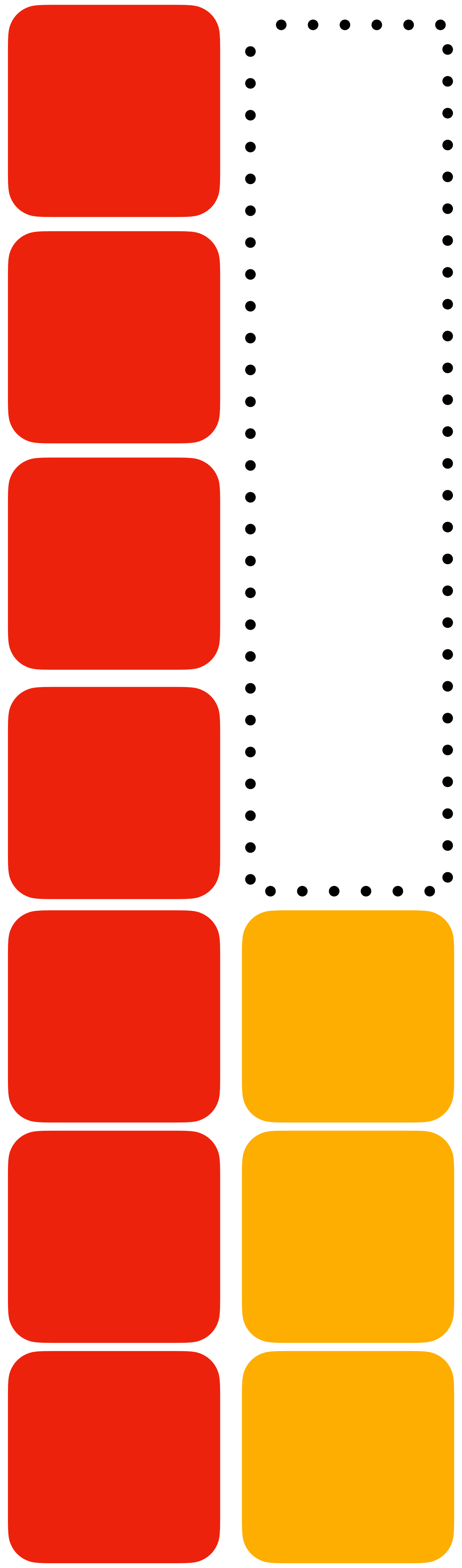


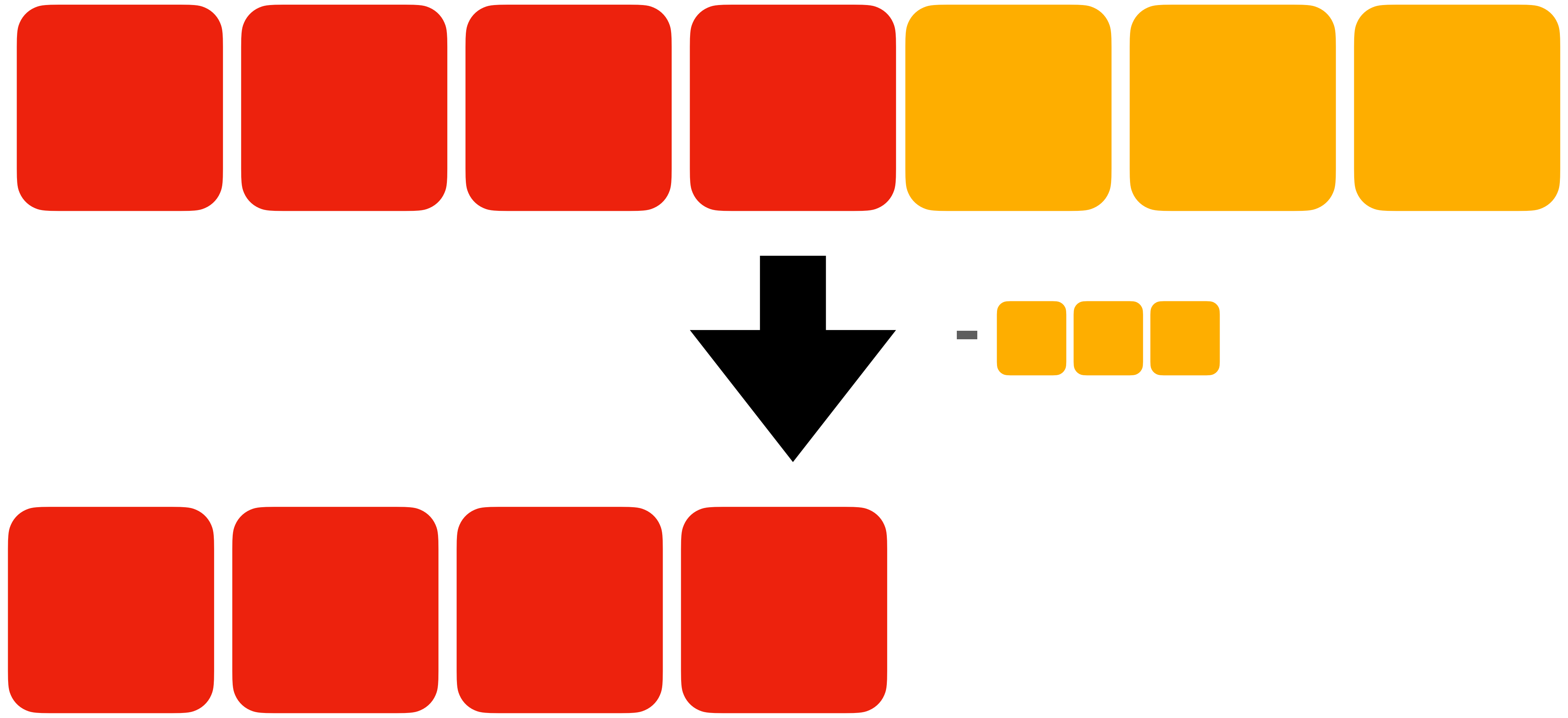
$$f(x, y) = x + y$$

$$f_a(x) = x + a$$

Subtraction

Take away / compare





$$f_a(x) = x - a = f_{-a}(x)$$

Mathematics (NE)

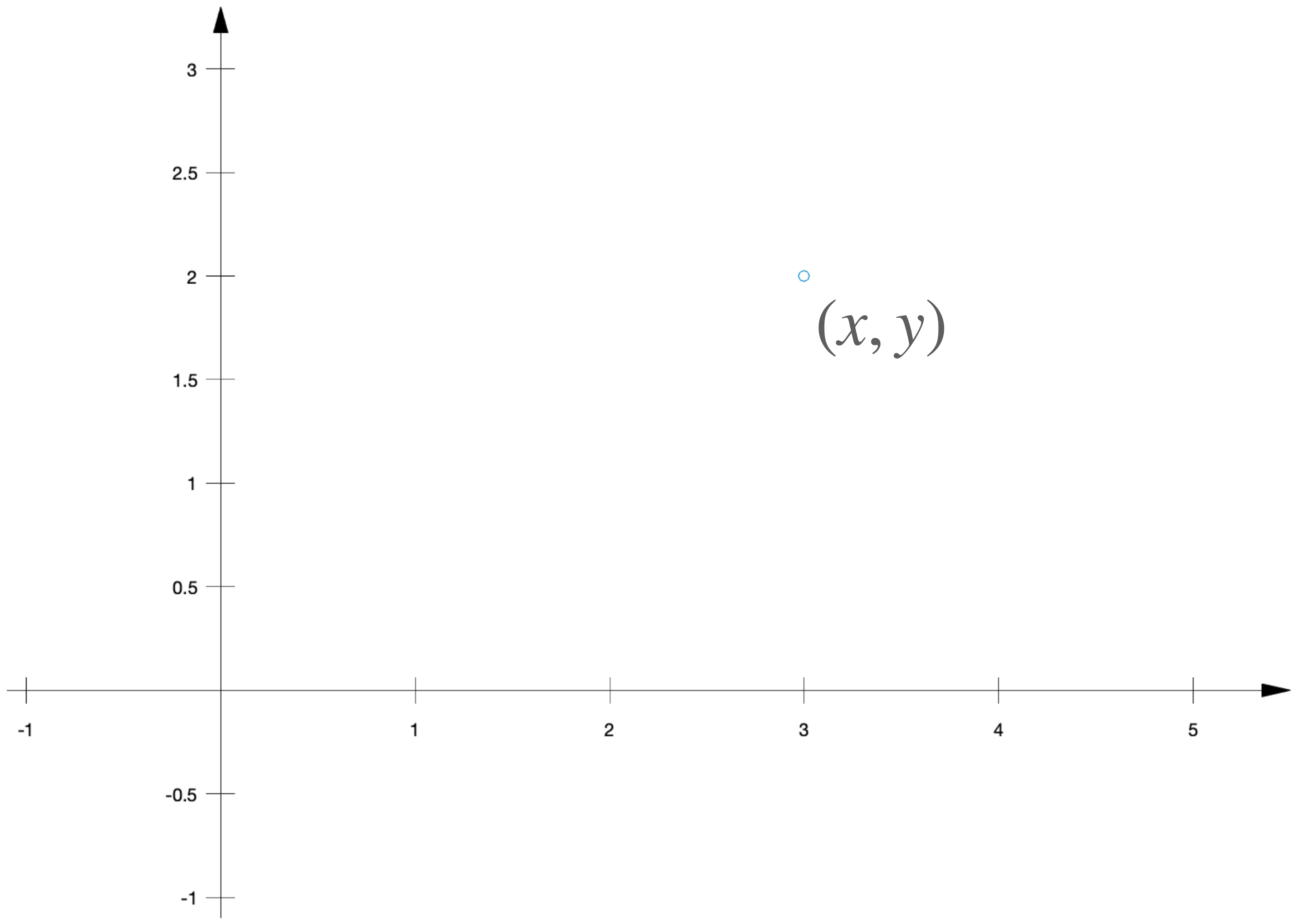
An abstract and general science for problem solving and methods development

… and symbol system development

345342

$$f(x)$$

$$\int_a^b f(x)dx$$



$$\frac{a}{b}$$

Symbol systems

1. a set of arbitrary physical tokens that are
2. manipulated on the basis of "explicit rules" that are
3. likewise physical tokens and strings of tokens. The rule-governed symbol-token manipulation is based
4. purely on the shape of the symbol tokens (not their "meaning"), i.e., it is purely syntactic, and consists of
5. "rulefully combining" and recombining symbol tokens. There are
6. primitive atomic symbol tokens and
7. composite symbol-token strings. The entire system and all its parts–the atomic tokens, the composite tokens, the syntactic manipulations both actual and possible and the rules– are all
8. "semantically interpretable:" The syntax can be systematically assigned a meaning (e.g., as standing for objects, as describing states of affairs) (Harnad, 1990, p. 336).

$$a, = , + , \cdot \qquad \rightarrow \qquad \frac{a}{b}, = , + , \cdot$$

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ab = cb$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{7}{4}$$

$$\frac{1}{3}$$

$$\frac{a/b}{a}$$

$$\frac{0.5}{6.7}$$

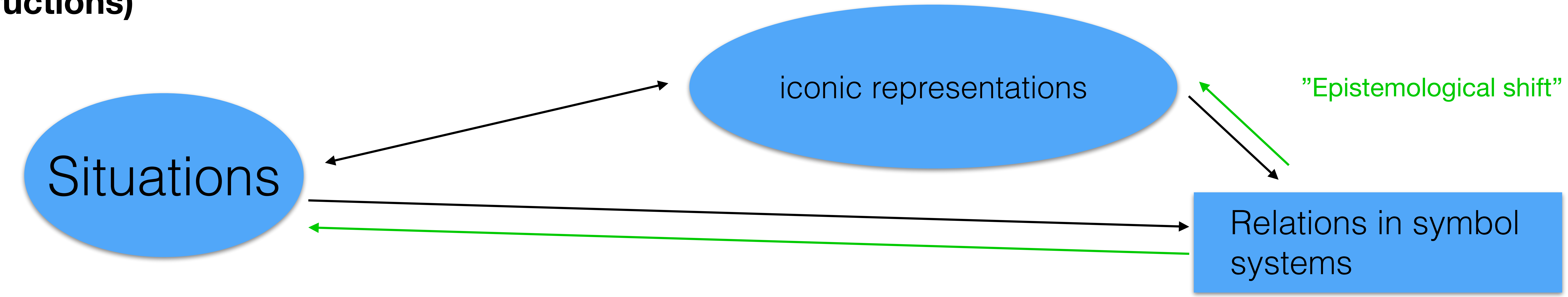
$$\frac{\cos(x)}{\sin(x)}$$

$$\frac{\sqrt{3}}{2}$$

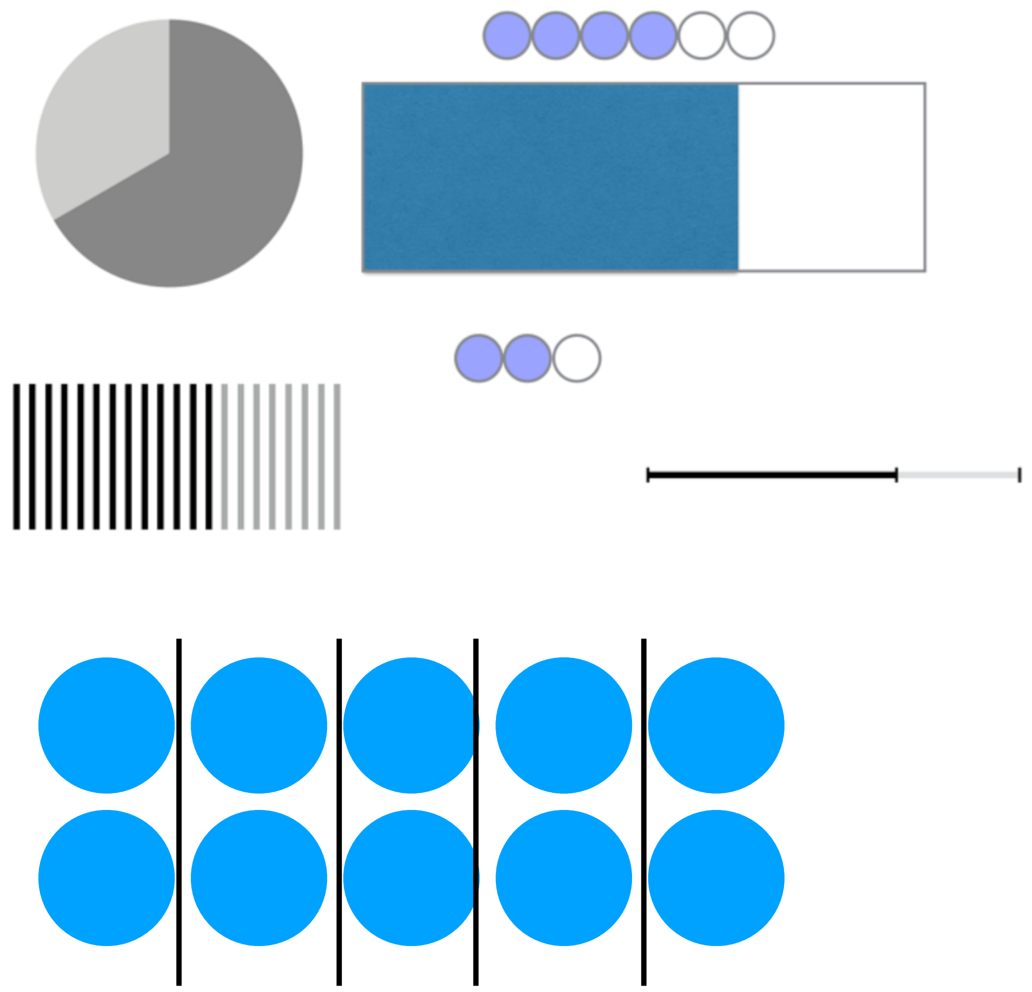
$$\frac{x^2+1}{2x^3-4x}$$

$$\frac{\sum_{n=0}^{\infty} a_n X^n}{\sum_{n=0}^{\infty} b_n X^n}$$

Fractions
(Quiotient constructions)



Part-whole
Division (partitive, equal sharing)
Measuring
Linearity
Rate
Ratio
Proportionality



$$\frac{2}{3}$$

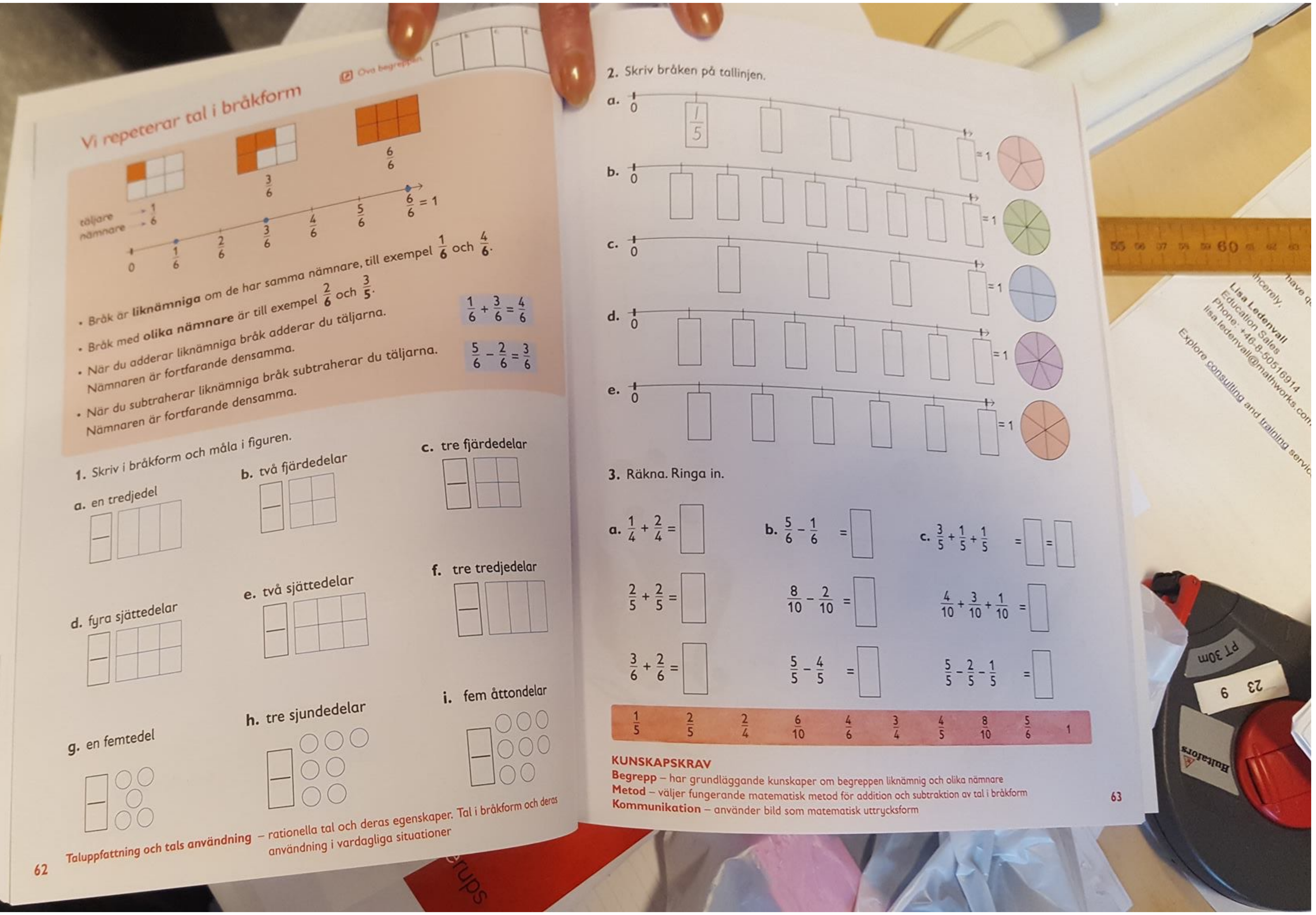
$$\frac{10}{5}$$

"How many 2's are there in 10?"

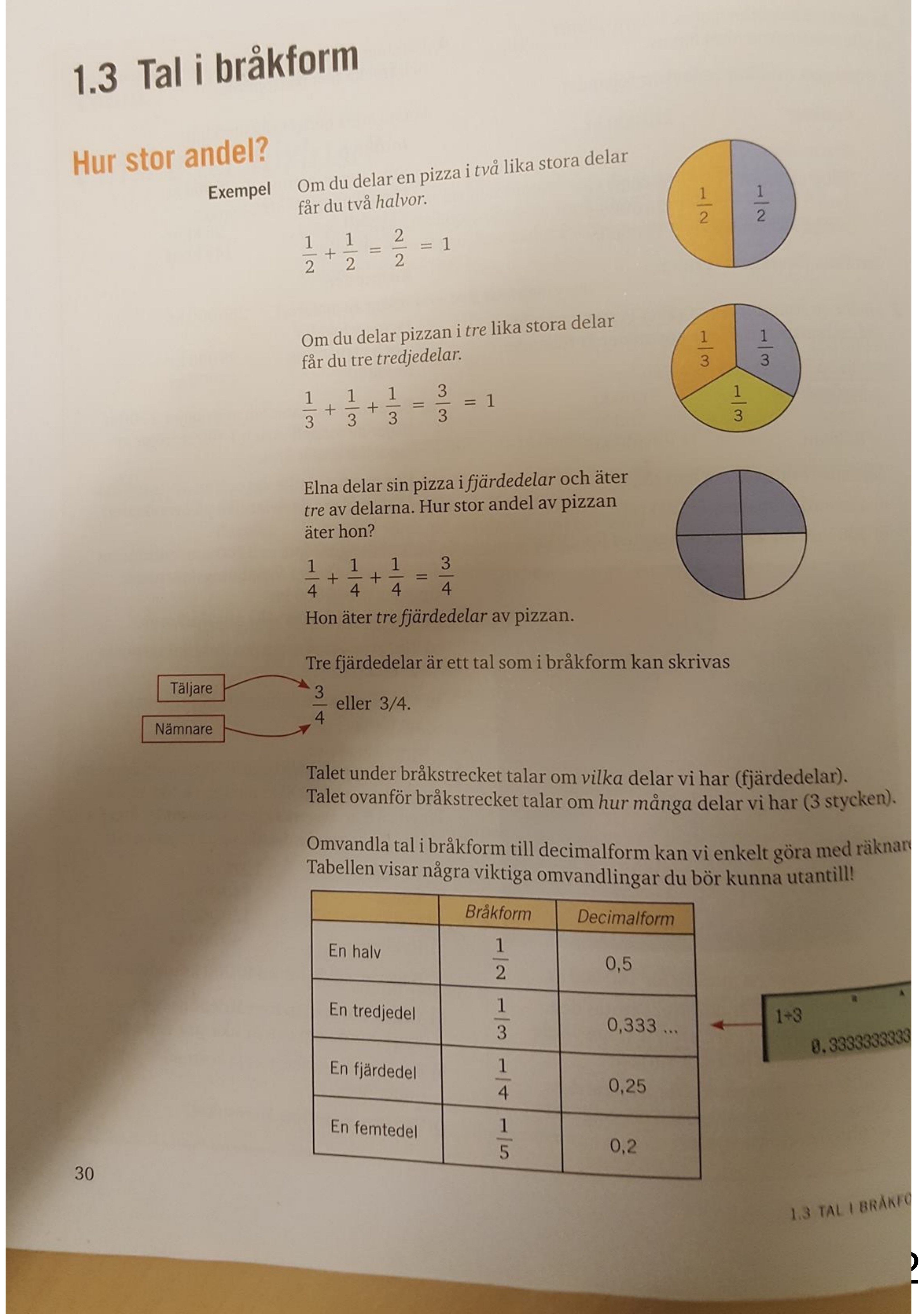
$\frac{a}{b}$ is a symbol c such that $a = bc$

Ahl & Helenius 2018, 2021
Vergnaud, 2009

Grade 5 ("We repeat fractions")



Upper secondary grade 1 ("grade 10")



38. Division – delningsdivision

Charlie, Isa och Liam delar 12 godisbitar lika mellan sig.
”En till Charlie, en till Isa, en till Liam, en till Charlie, en till Isa, en till Liam …”

C I L C I L C I L C I L

Alla får fyra godisbitar var.

Öva begreppen.
Lyssna på berättelsen.

1. Dela godisbitarna lika mellan Charlie (C), Isa (I) och Liam (L). Hur många godisbitar får de var?

a. C I L

Svar: 1 godisbit

b.

Svar: godisbitar

39. Division – innehållsdivision

Film
Lyssna på berättelsen.

Charlie lägger 12 bullar i påsar. Han lägger 4 bullar i varje påse.

Hur många påsar behöver Charlie?

Svar: 3 påsar

1. Du lägger 18 bullar i påsar. Hur många påsar behöver du?

a. Det ska vara 3 bullar i varje påse.

Svar: påsar

b. Det ska vara 6 bullar i varje påse.

Svar:

40. Att skriva division

Lyssna på berättelsen.
Film

Det finns sex glasskulor.
I varje bägare ska det vara 2 kulor glass. Hur många bägare behövs det?

Divisionen kan skrivas

$\frac{6}{2} = 3$ $\frac{\text{täljare}}{\text{nämnare}} = \text{kvot}$

eller

$6/2 = 3$

täljare/nämnare = kvot

Du säger: Sex delat med två är lika med tre, eller sex dividerat med två är lika med tre.

1. Dividera.

a. $\frac{8}{2} = $ ____

b. $\frac{9}{3} = $ ____

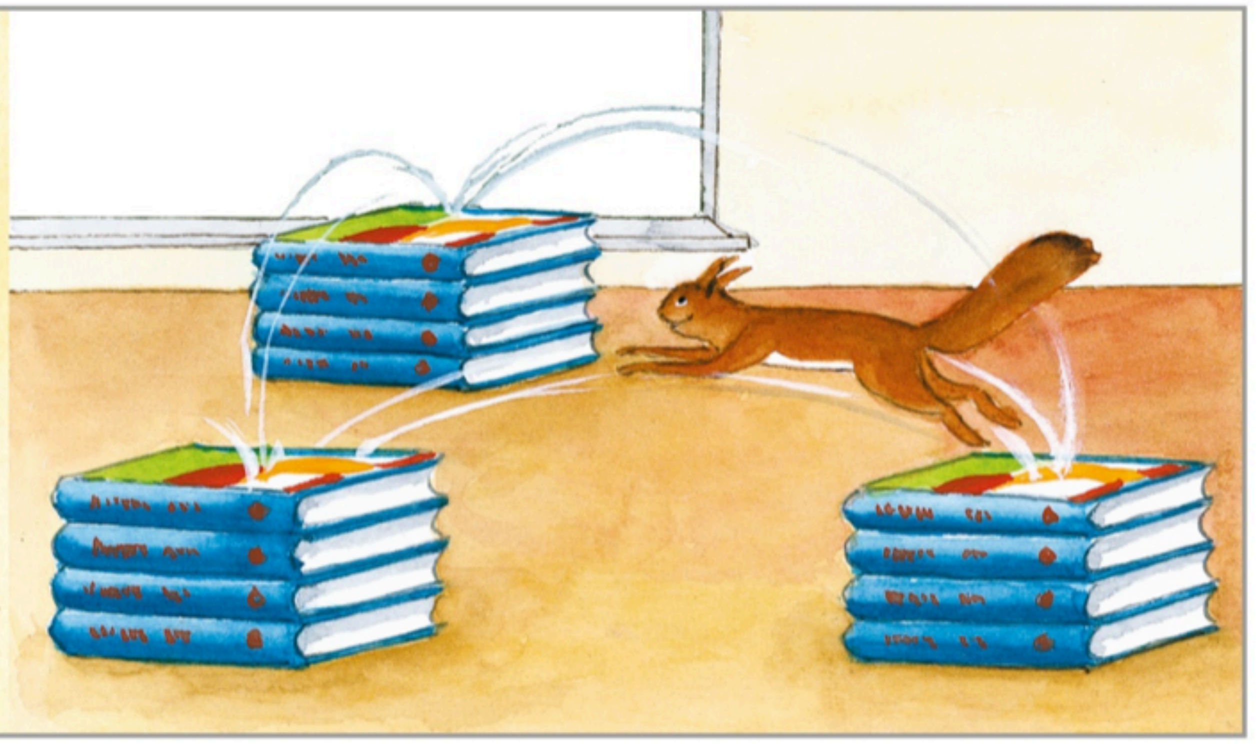
41. Sambandet mellan division och multiplikation

Film
Lyssna på berättelsen.

Det finns 12 böcker. Hur många högar får du, om du lägger 4 böcker i varje hög?

$\frac{12}{4} = 3$

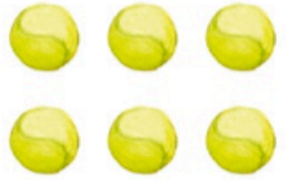
Du kan kontrollera division med multiplikation.
$3 \cdot 4 = 12$



1. Dividera. Kontrollera division med multiplikation.

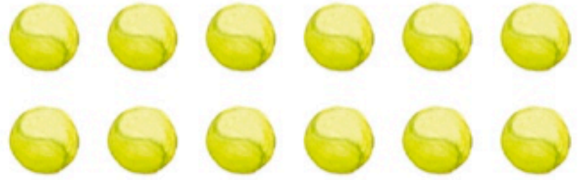
a. $\frac{8}{2} = $ 4

4 · 2 = 8



b. $\frac{6}{3} = $ ____

____ · ____ = 6



c. $\frac{12}{6} = $ ____

____ · ____ = ____

Grade	Mathematical content	Concept	Procedure	Connection
3	division, equal sharing	☺		
	division, equal grouping	☺ ⟶ ♥		
	division, equal grouping	☺ ⟶ π		
	multiplication and division	☺ ⟶ π		♥ ⟶ π
	division, equal grouping	☺ ⟶ π		
	proportionality	☺ ⟶ π		
	fractions, geometric part whole	♥		
	fractions, geometric part whole	♥ ⟶ π		
	fractions, geometric part whole	♥ ⟶ π		
	fractions, one whole	♥ ⟶ π		
4	division, equal grouping and sharing	☺+♥ ⟶ π		
	division, short division		♥ ⟶ π	
	division, short division		⟶ π	
	fractions, equal fractios	♥ ⟶ π		
5	fractions, mixed fractions	♥ ⟶ π	⟶ π	
	fractions, reducing fractions	♥ ⟶ π	⟶ π	
	fractions, reducing to lowest term		⟶ π	
	dividing fraction with whole number		⟶ π	☺+♥ ⟶ π
	multiply fraction with whole number		⟶ π	☺+♥ ⟶ π
	fractions, fraction of numbers	♥ ⟶ π	⟶ π	
6	fractions, geo. part whole, mixed	♥ ⟶ π		
	fractions, convert to and from mixed		⟶ π	
	fractions, reducing, reducing to lowest term	♥ ⟶ π	⟶ π	
	fractions, expanding fractions	♥ ⟶ π	⟶ π	

Ahl & Helenius, 2021

Grade	Mathematical content	Concept	Procedure	Connection
7	fractions, geometric part whole	♥ → π		
	fractions, geometric part hole bigger than one	♥ → π		
	fractions, size of fractions	♥ → π	♥ → π	
	fractions, equal fractions	♥ → π		
	fractions, reduce fractions	♥ → π	→ π	
	fractions, expanding fractions	♥ → π	→ π	
	fractions, fraction of number	☺+♥ → π		
	fractions, geometric part whole	♥ → π		
	fractions, fractions bigger than one	☺+♥ → π		
	fractions, equal fractions	π , ♥ → π		
8	fractions, part whole of numbers	☺+♥ → π		
	fractions, equal fractions	♥ → π	→ π	
	multiply a fraction with a whole number		π	☺+♥ → π
	multiply fractions	♥ → π	→ π	
	multiply fractions		→ π	
	fractions	♥ → π		
	fractions, expand and reduce	♥ → π	→ π	
	fractions, reduce fractions with variables*	π	→ π	
	fractions, division of fractions*			☺+♥ → π
	fractions, division of fractions, inverse*	π, π → ♥	π	π, ♥ → π
	fractions, division of fractions with variables*	π	π	
9	fractions, comparing fractions	♥ → π		
	fractions, equal fractions, reducing, expanding	♥ → π	→ π	
	fractions, multiply fractions		→ π	♥ → π

* From the ”red course” with advanced material that not all student will nessecarily encounter.

Ahl & Helenius, 2021

Polysemy

When one word has several but related meanings

When one concept has several but related meanings

Part-whole
Division (partitive, equal sharing)
Measuring
Linearity
Rate
Ratio
Proportionality

$$\frac{a}{b}$$

On Proof and Progress in Mathematics

WILLIAM P. THURSTON

People have very different ways of understanding particular pieces of mathematics. To illustrate this, it is best to take an example that practicing mathematicians understand in multiple ways, but that we see our students struggling with. The derivative of a function fits well. The derivative can be thought of as:

(1) Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.
(2) Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, the derivative of $f \circ g$ is $f' \circ g * g'$, etc.
(3) Logical: $f'(x) = d$ if and only if for every ε there is a δ such that when $0 < |\Delta x| < \delta$,

$$\left| \frac{f(x+\Delta x) - f(x)}{\Delta x} - d \right| < \delta.$$

(4) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.
(5) Rate: the instantaneous speed of $f(t)$, when t is time.
(6) Approximation: The derivative of a function is the best linear approximation to the function near a point.
(7) Microscopic: The derivative of a function is the limit of what you get by looking at in under a microscope of higher and higher power.

The list continues; there is no reason for it ever to stop.

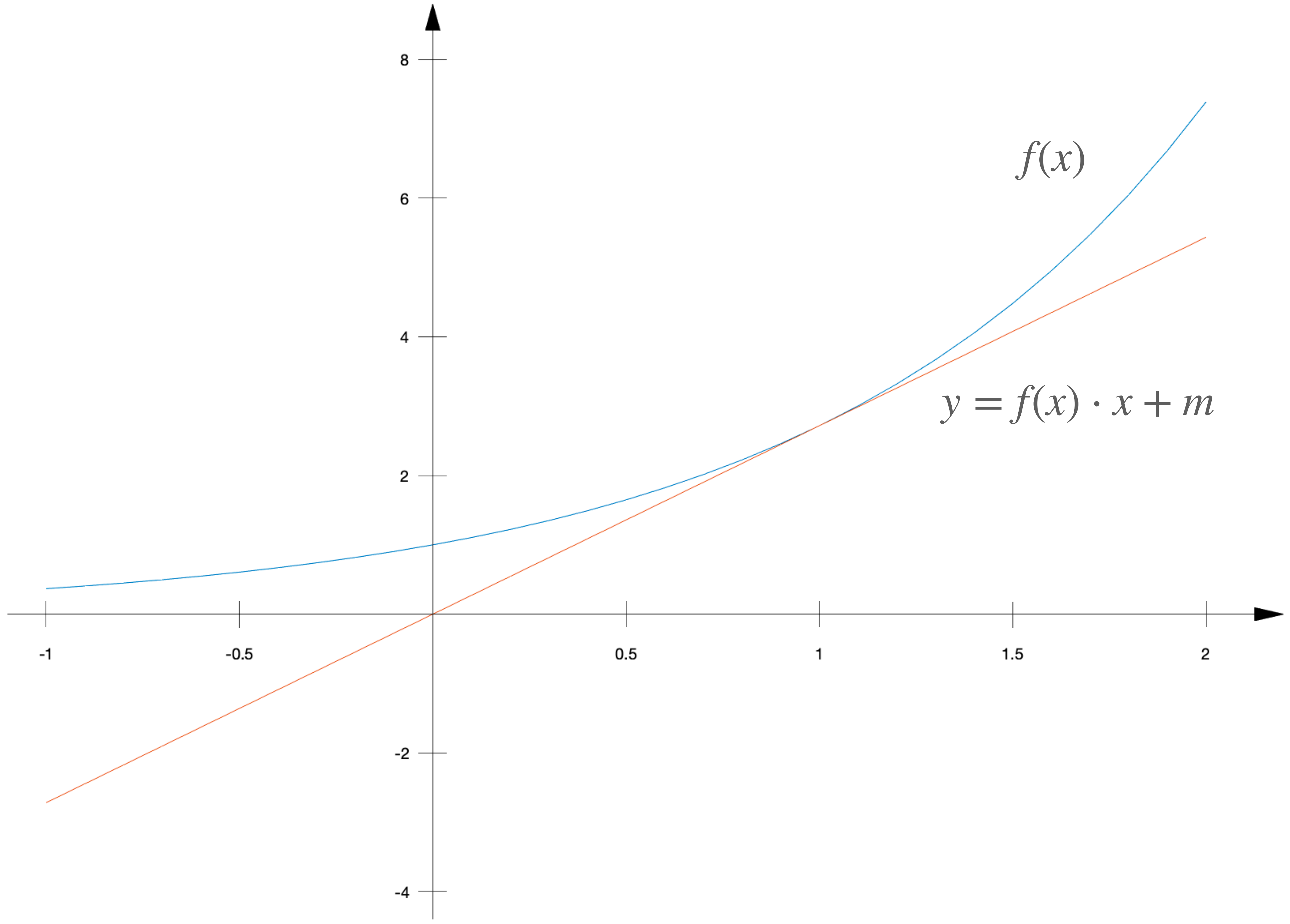
37. The derivative of a real-valued function f in a domain D is the Lagrangian section of the cotangent bundle $T^*(D)$ that gives the connection form for the unique flat connection on the trivial $\mathbf{R}$-bundle $D \times \mathbf{R}$ for which the graph of f is parallel.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

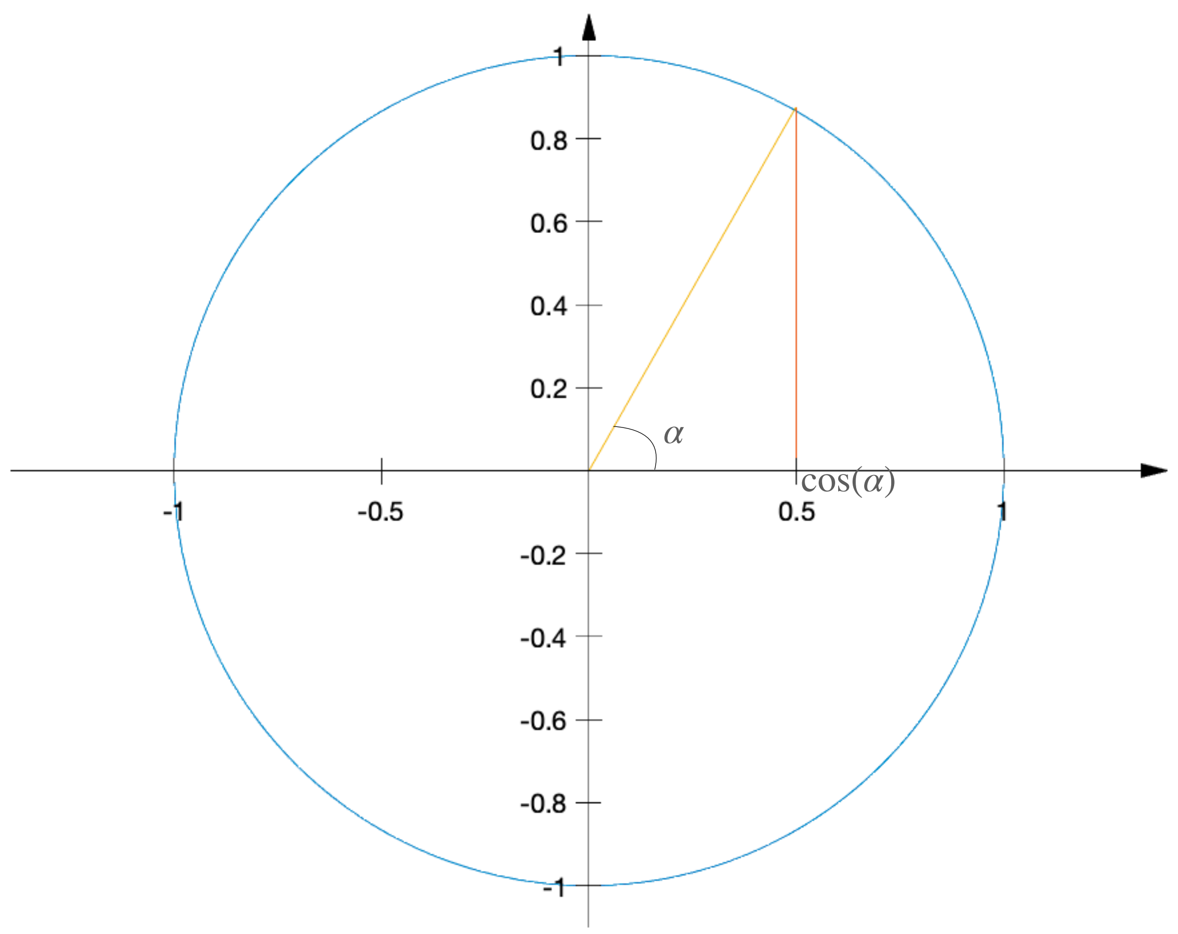
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n, \;\; e^x = ?$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

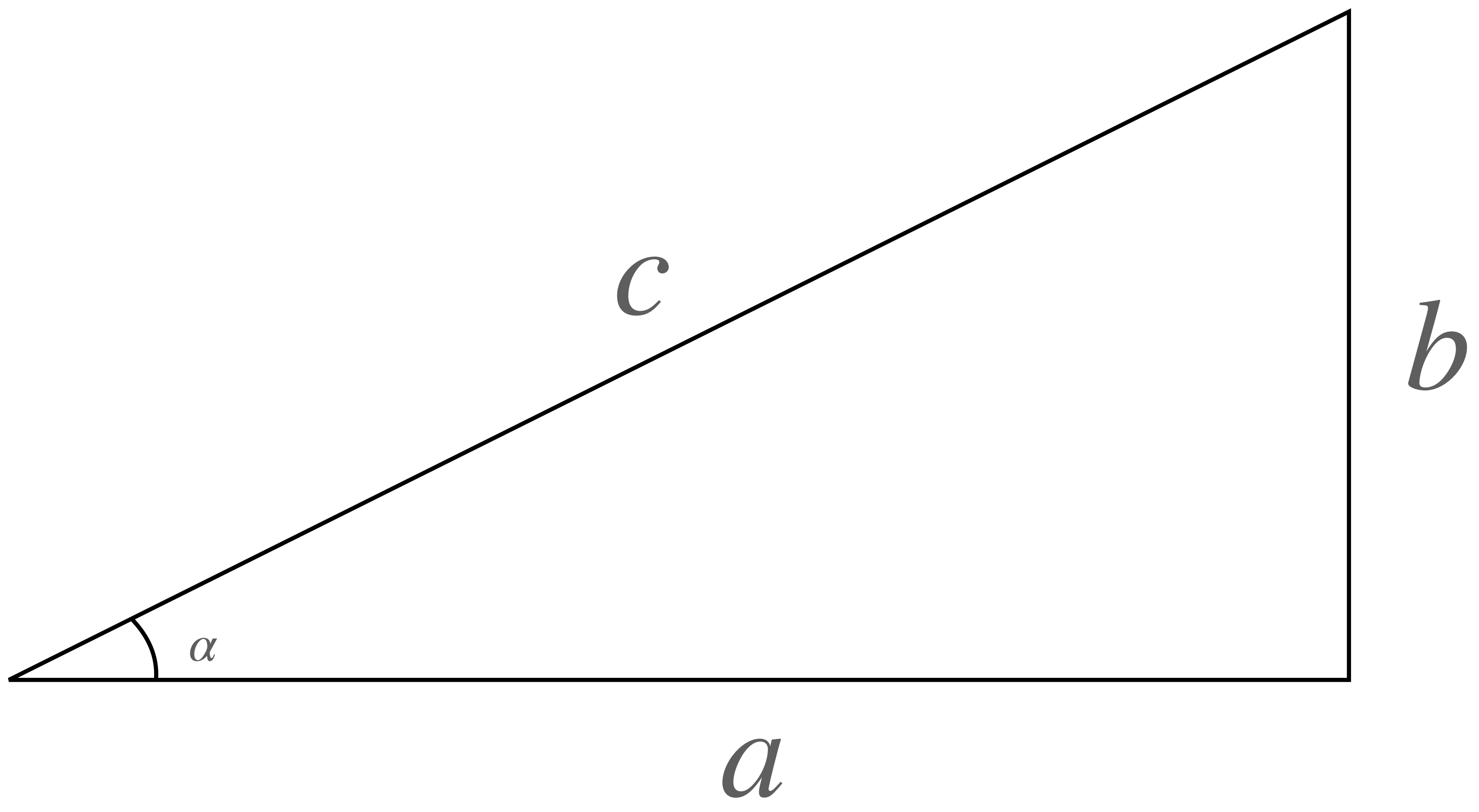
$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$



$$e^x = y : y' = y, \;\; y(0) = 1$$



$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$$



$$\cos(\alpha) = a/c$$

Consequences for teaching?

Thank you for listening