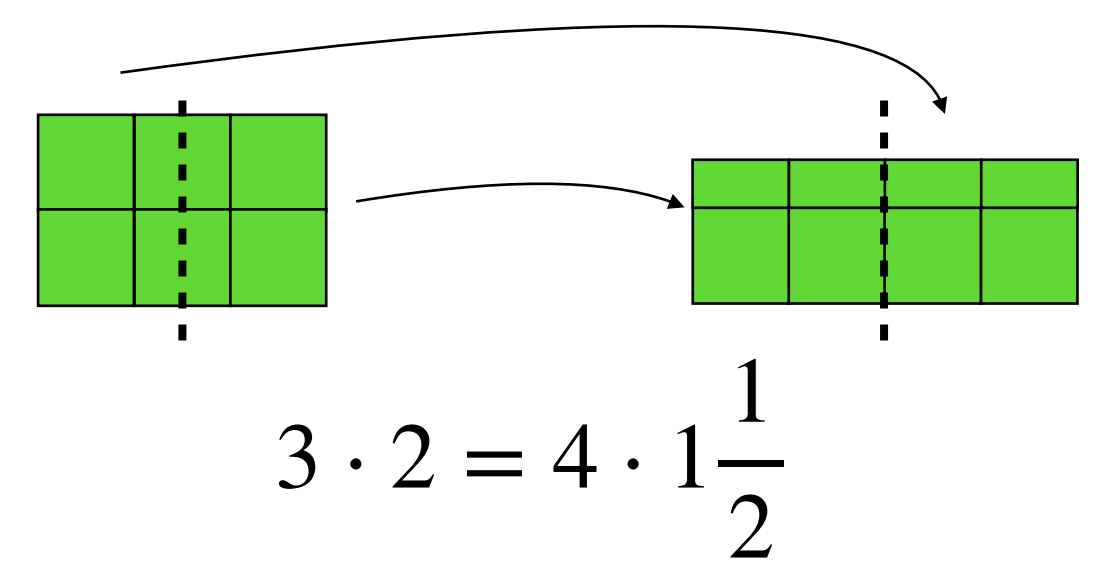
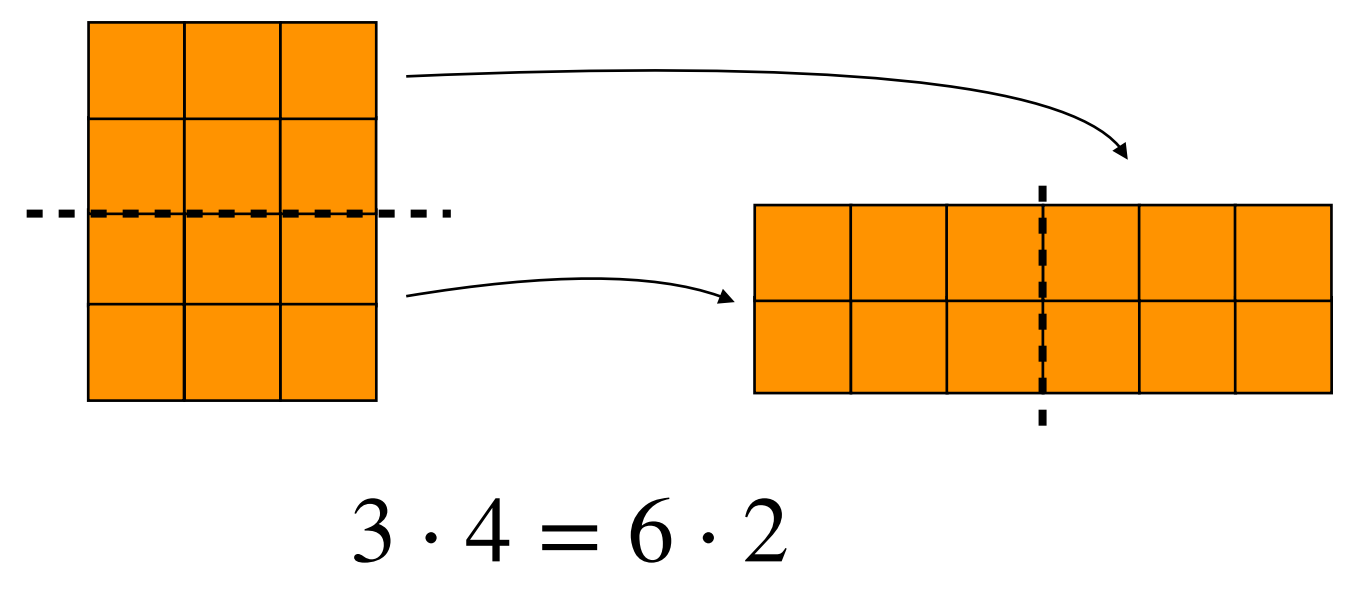
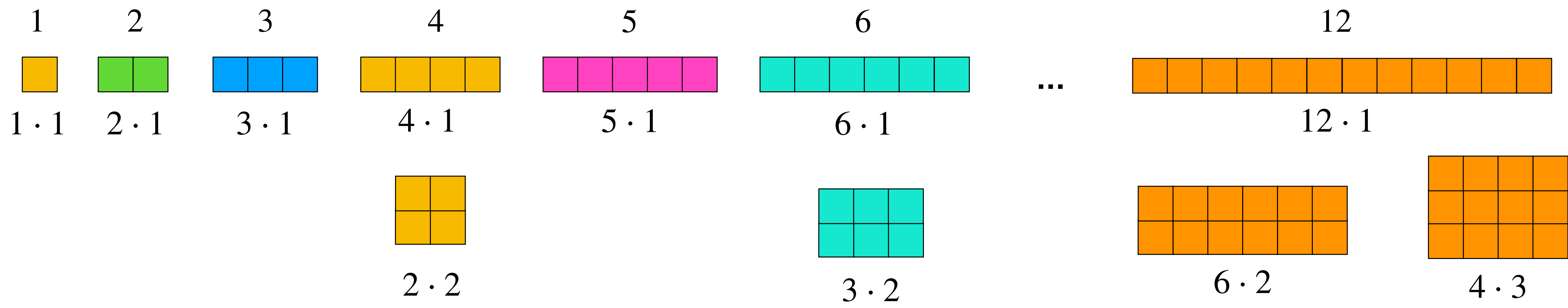
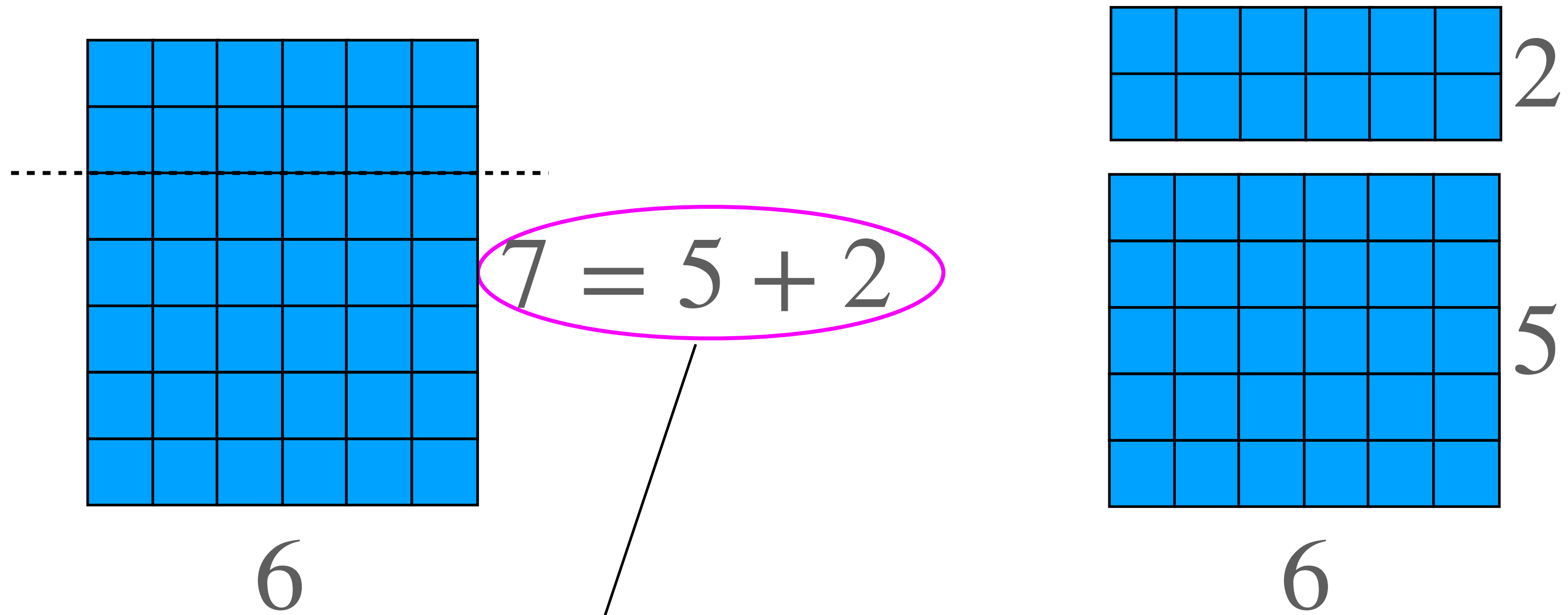


# The many meanings of mathematical concepts

Ola Helenius 221102

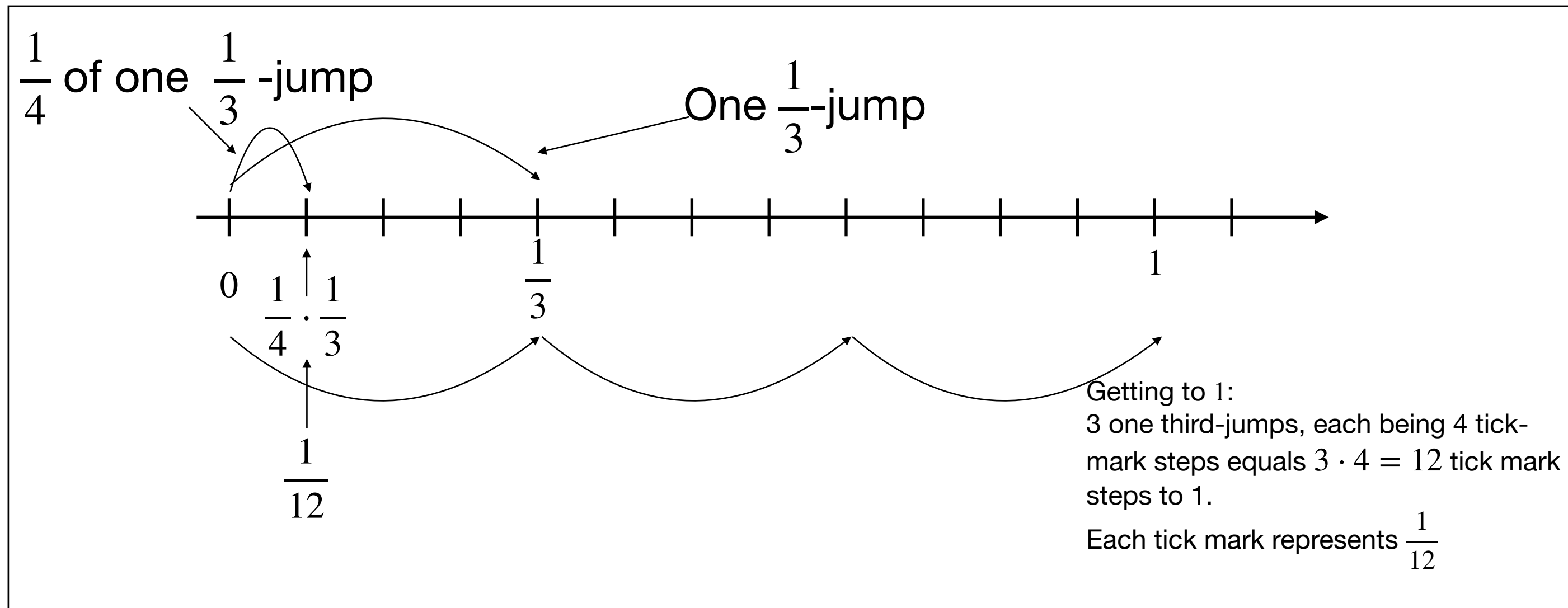
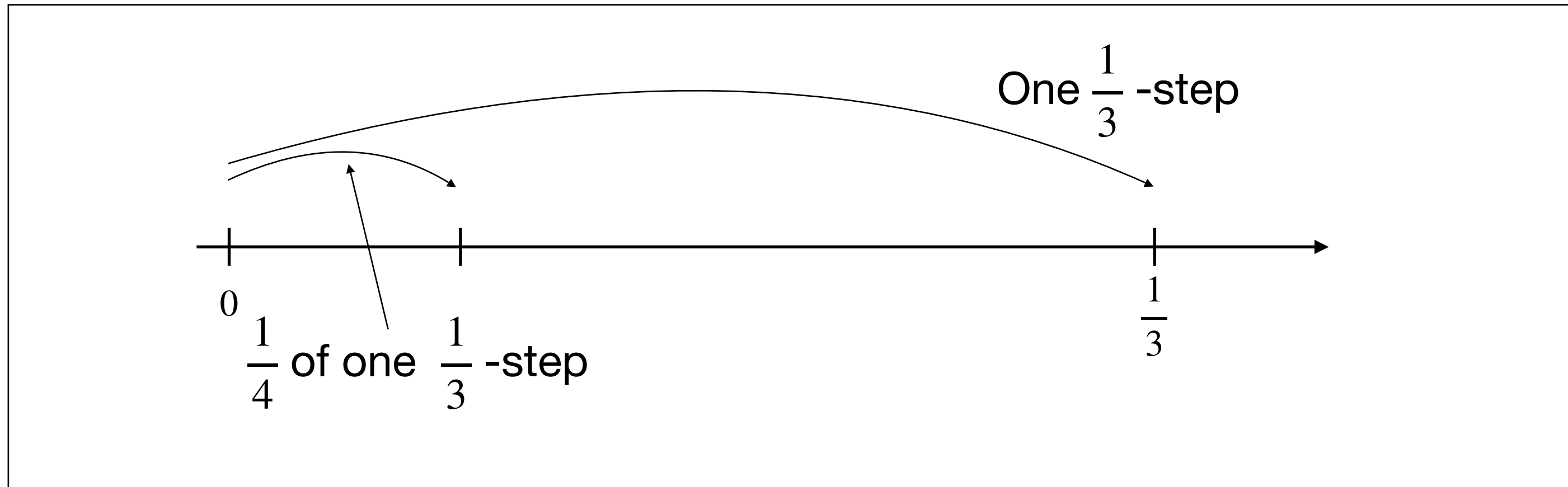






$$6 \cdot 7 = 6 \cdot (5 + 2) = 6 \cdot 5 + 6 \cdot 2$$

$$6 \cdot 7 = 6 \cdot (5 + 2) = 6 \cdot 5 + 6 \cdot 2$$

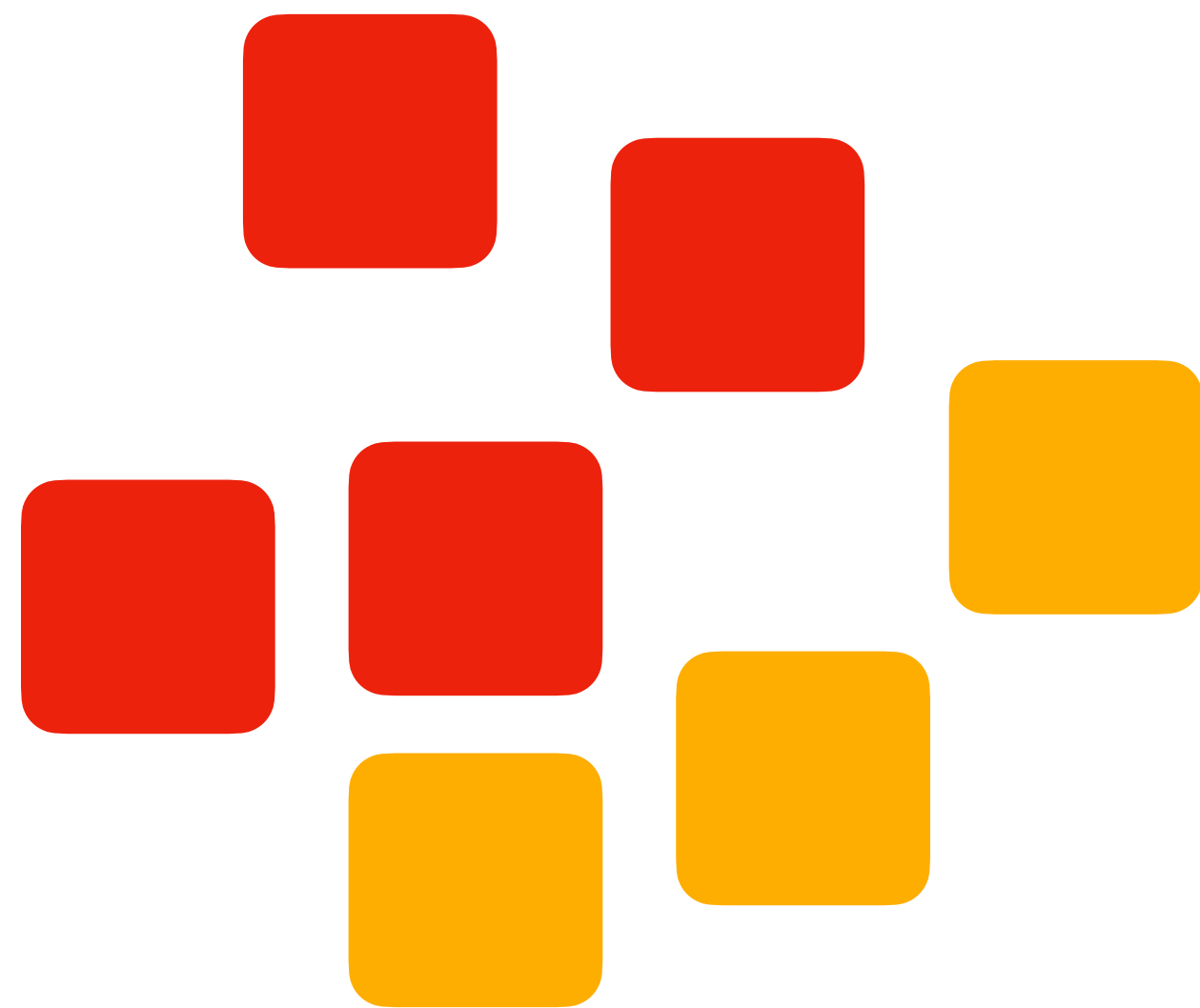


	Little Bear	Big Bear	Mama Bear
waterbottles	1	$1 \cdot 2 = 2$	$1 \cdot 6 = 6$
almonds	5	10	30
bisquits	2	4	12
apples	$\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2}$	1	$3 = 6 \cdot \frac{1}{2} = 3 \cdot 1$
nuts	5	10	$30 = 6 \cdot 5 = 3 \cdot 10$
grapes	3	$6 = 2 \cdot 3 = \frac{1}{3} \cdot 18$	18
buns	$\frac{1}{2}$	$1 = 2 \cdot \frac{1}{2} = \frac{1}{3} \cdot 3$	3

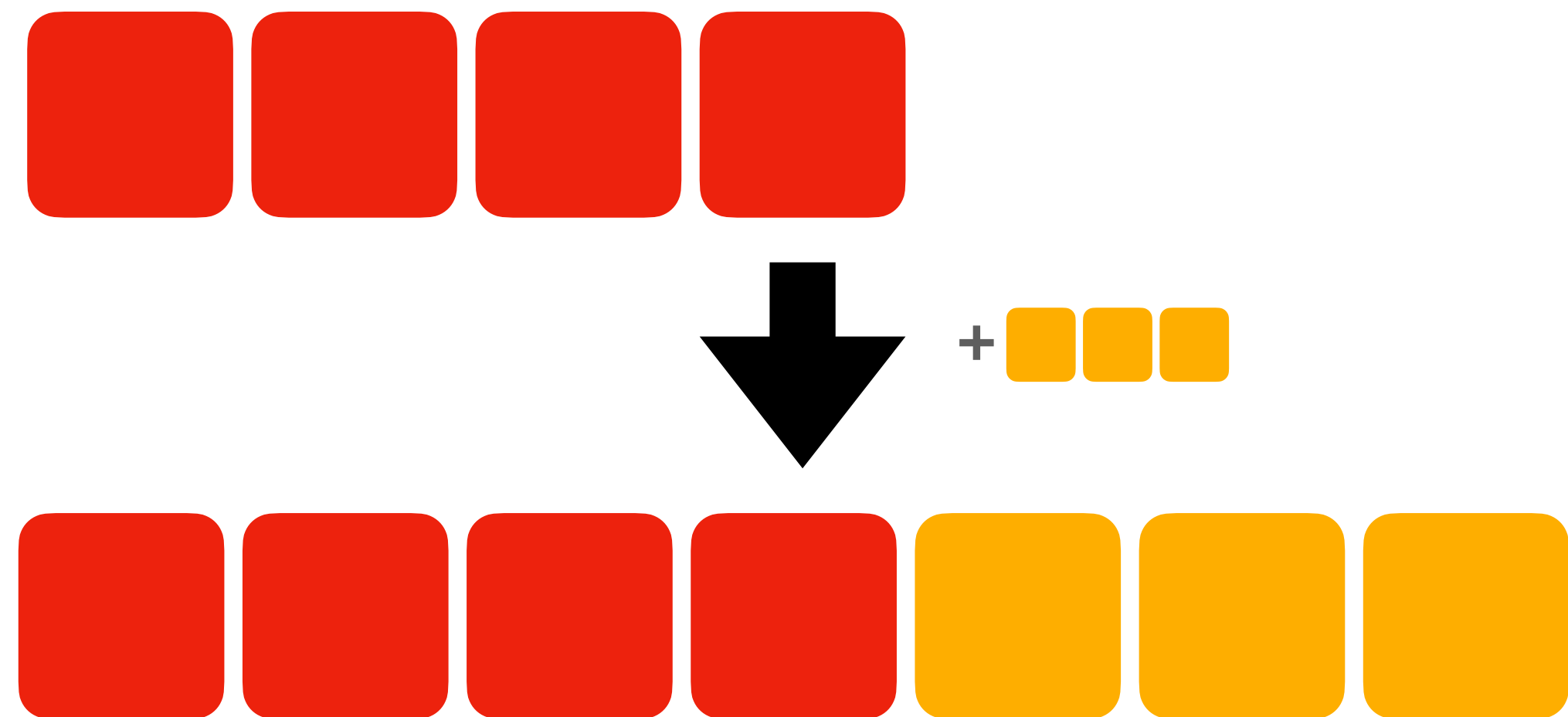
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# Situations and iconic schematic imagery

Addition - "lägga ihop" / "lägga till"



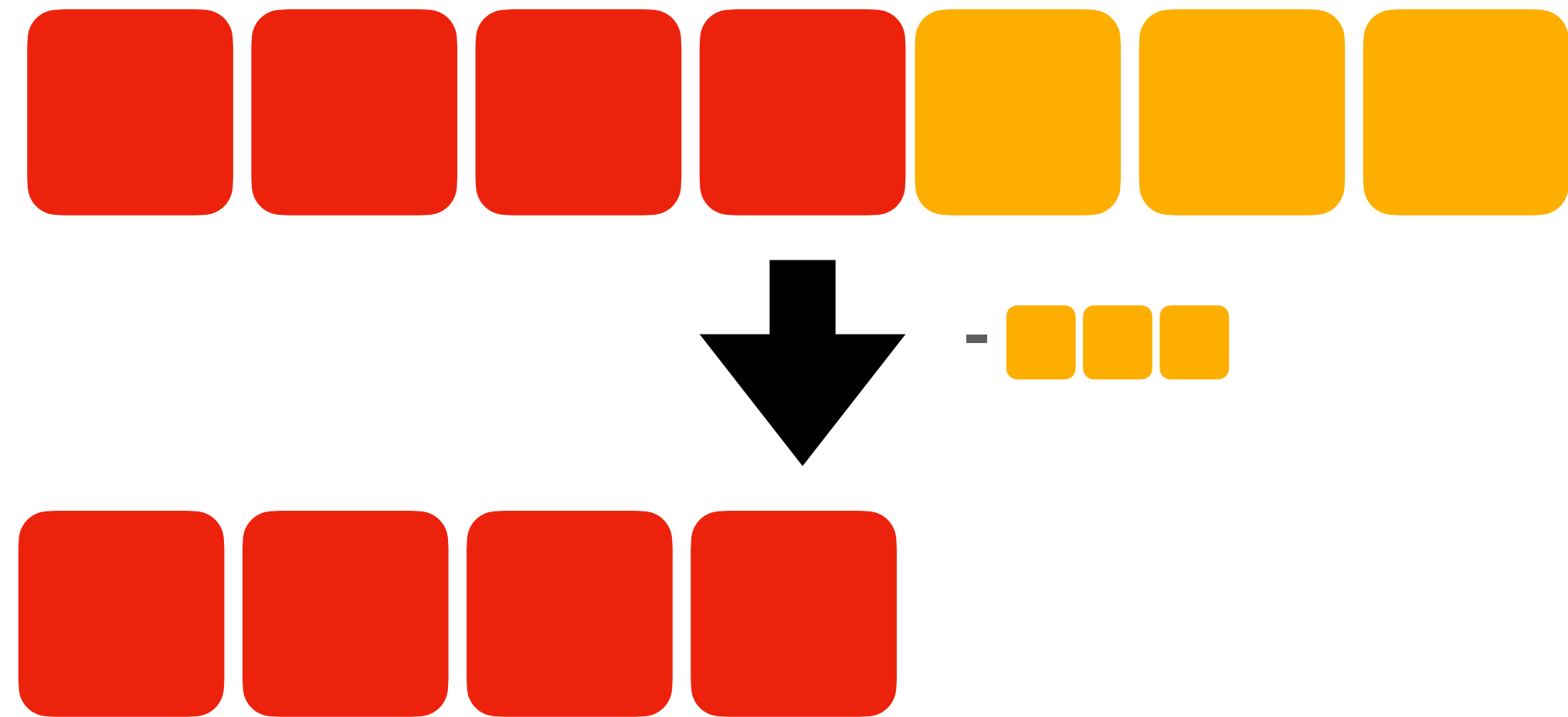
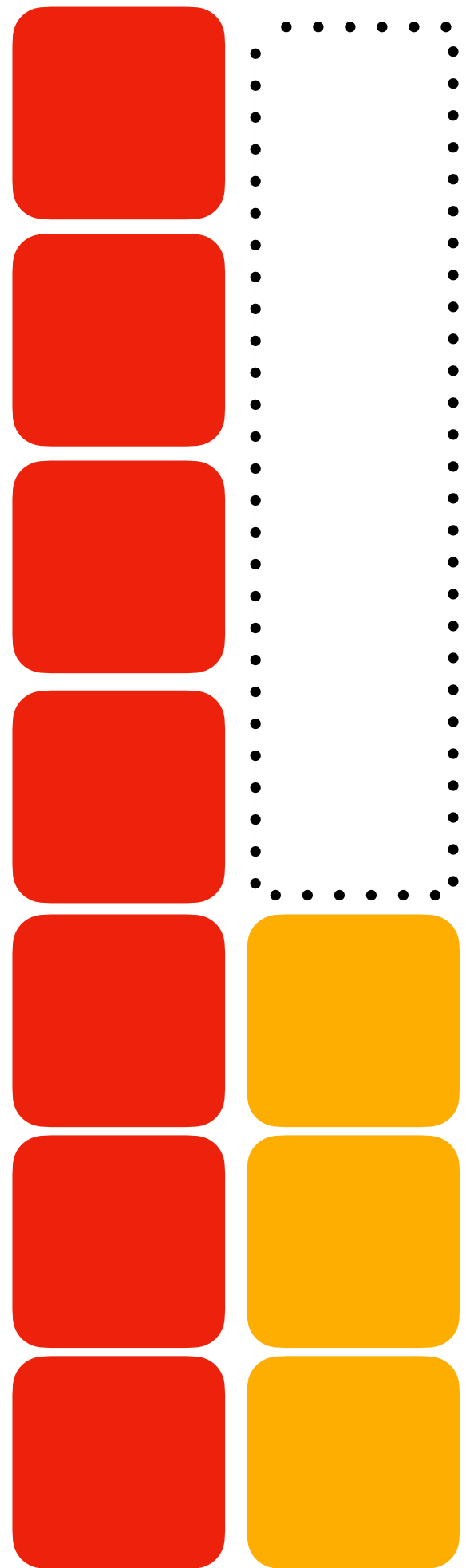
$$f(x, y) = x + y$$



$$f_a(x) = x + a$$

# Subtraction

Take away / compare



$$f_a(x) = x - a = f_{-a}(x)$$



# Mathematics (NE)

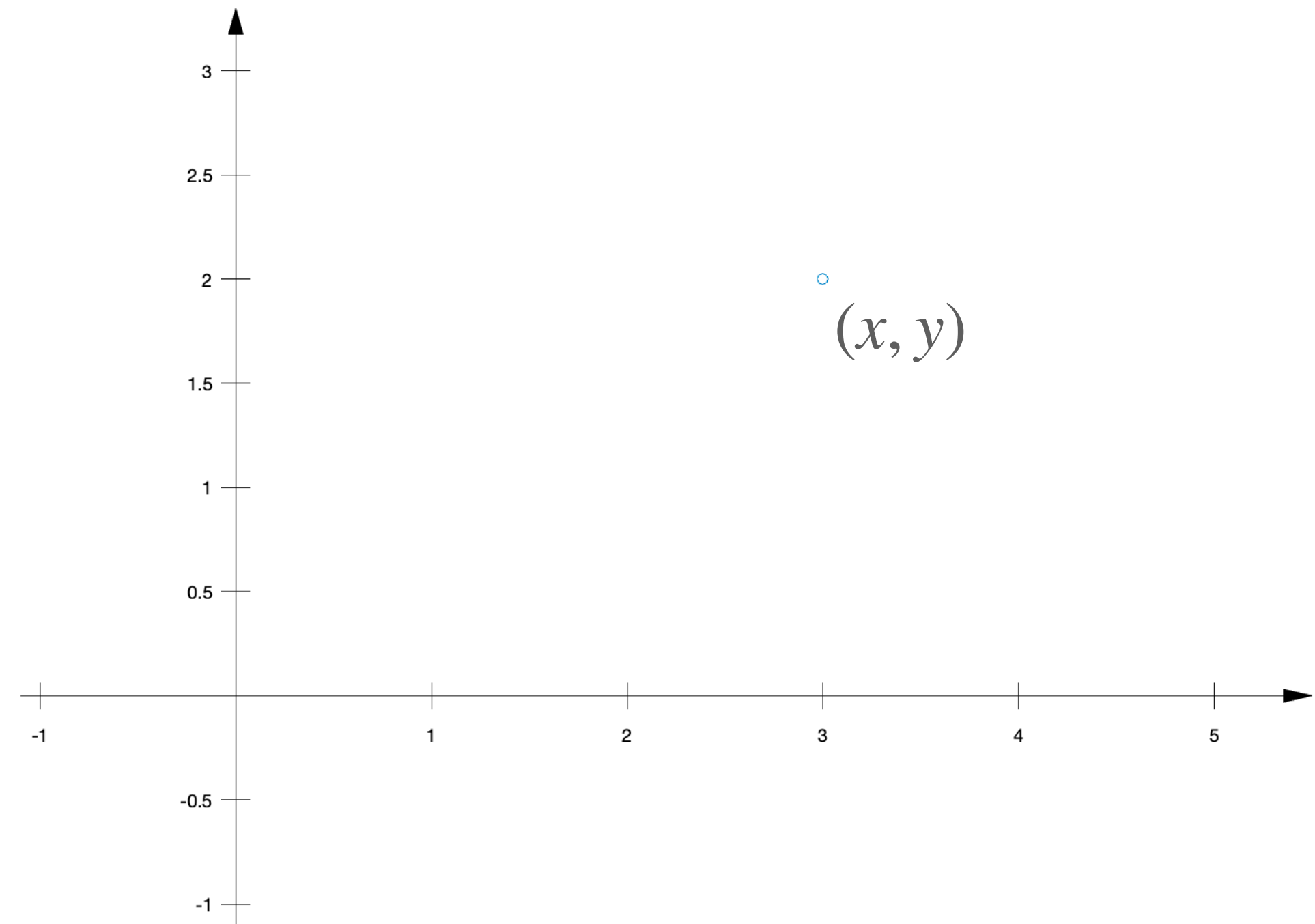
**An abstract and general science for problem solving and methods development**

**... and symbol system development**

345342

$f(x)$

$$\int_a^b f(x) dx$$



$$\frac{a}{b}$$

# Symbol systems

- 1 a set of arbitrary physical tokens that are
- 2 manipulated on the basis of "explicit rules" that are
- 3 likewise physical tokens and strings of tokens. The rule-governed  
symbol-token manipulation is based
- 4 purely on the shape of the symbol tokens (not their "meaning"), i.e., it  
is purely syntactic, and consists of
- 5 "rulefully combining" and recombining symbol tokens. There are
- 6 primitive atomic symbol tokens and
- 7 composite symbol-token strings. The entire system and all its parts—the  
atomic tokens, the composite tokens, the syntactic manipulations both  
actual and possible and the rules— are all
- 8 "semantically interpretable:" The syntax can be systematically assigned  
a meaning (e.g., as standing for objects, as describing states of affairs)  
(Harnad, 1990, p. 336).

$$a, =, +, \cdot$$

→

$$\frac{a}{b}, =, +, \cdot$$

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ab = cd$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{7}{4}$$

$$\frac{1}{3}$$

$$\frac{a/b}{a}$$

$$\frac{0.5}{6.7}$$

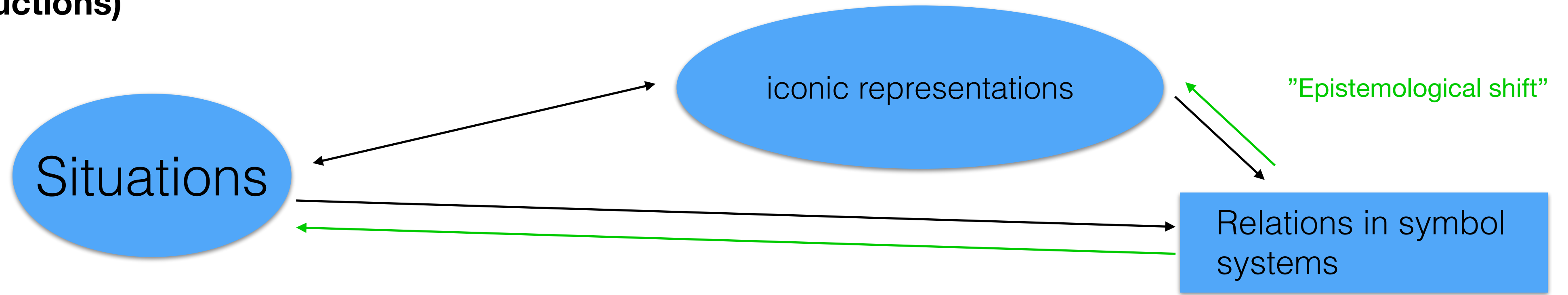
$$\frac{\cos(x)}{\sin(x)}$$

$$\frac{\sqrt{3}}{2}$$

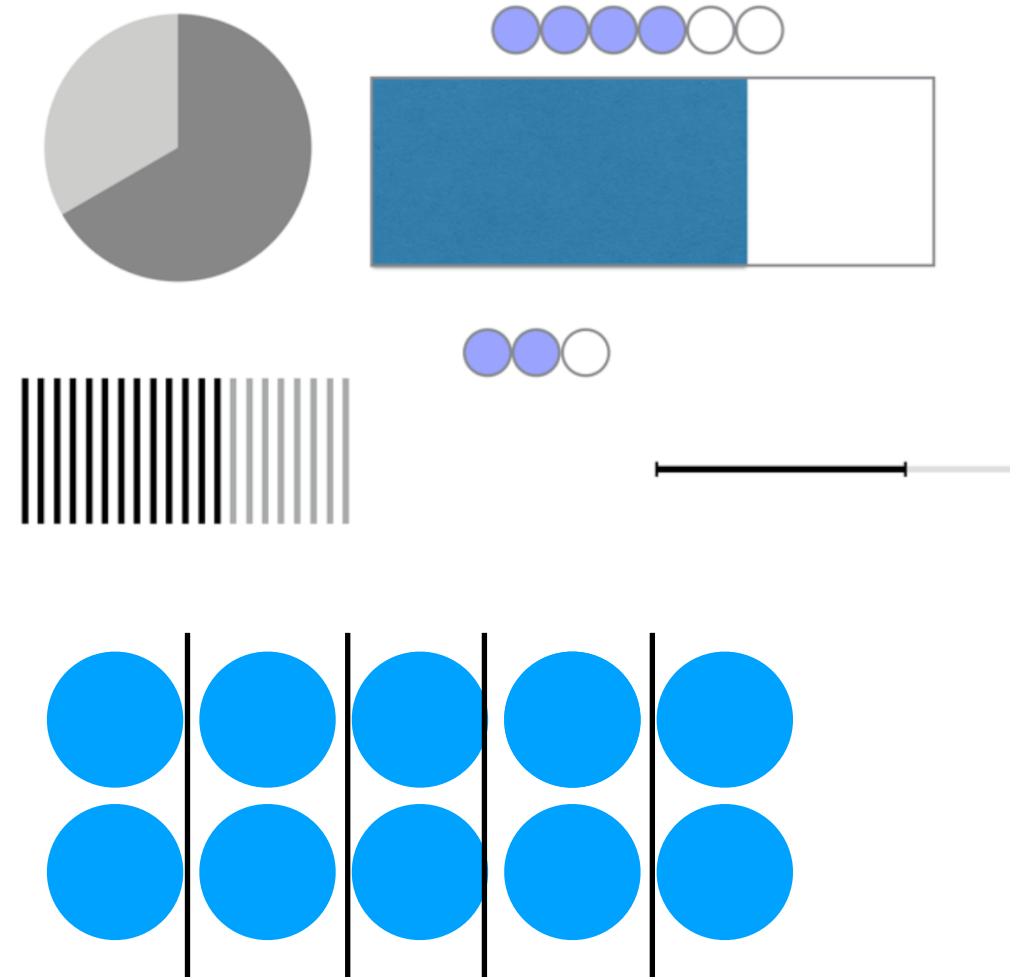
$$\frac{x^2 + 1}{2x^3 - 4x}$$

$$\frac{\sum_{n=0}^{\infty} a_n X^n}{\sum_{n=0}^{\infty} b_n X^n}$$

**Fractions  
(Quotient constructions)**



- Part-whole**
- Division (partitive, equal sharing)**
- Measuring**
- Linearity**
- Rate**
- Ratio**
- Proportionality**



$$\frac{2}{3}$$

$$\frac{10}{5}$$

**"How many 2's are there in 10?"**

$\frac{a}{b}$  is a symbol  $c$  such that  $a = bc$

# Grade 5 ("We repeat fractions")

**Vi repeterar tal i bråkform**

täljare → 1  
nämnamnare → 6

• Bråk är liknämninga om de har samma nämnare, till exempel  $\frac{1}{6}$  och  $\frac{4}{6}$ .

• Bråk med olika nämnare är till exempel  $\frac{2}{6}$  och  $\frac{3}{5}$ .

• När du adderar liknämninga bråk adderar du täljarna. Nämnaren är fortfarande densamma.

• När du subtraherar liknämninga bråk subtraherar du täljarna. Nämnaren är fortfarande densamma.

$\frac{1}{6} + \frac{3}{6} = \frac{4}{6}$

$\frac{5}{6} - \frac{2}{6} = \frac{3}{6}$

1. Skriv i bråkform och måla i figuren.

a. en tredjedel

b. två fjärdedelar

c. tre fjärdedelar

d. fyra sjättedelar

e. två sjättedelar

f. tre tredjedelar

g. en femtedel

h. tre sjundedelar

i. fem åttondelar

2. Skriv bråken på tallinjen.

a.  $\frac{1}{5}$

b.  $\frac{1}{3}$

c.  $\frac{1}{4}$

d.  $\frac{1}{5}$

e.  $\frac{1}{6}$

3. Räkna. Ringa in.

a.  $\frac{1}{4} + \frac{2}{4} = \square$

b.  $\frac{5}{6} - \frac{1}{6} = \square$

c.  $\frac{3}{5} + \frac{1}{5} + \frac{1}{5} = \square = \square$

$\frac{2}{5} + \frac{2}{5} = \square$

$\frac{8}{10} - \frac{2}{10} = \square$

$\frac{4}{10} + \frac{3}{10} + \frac{1}{10} = \square$

$\frac{3}{6} + \frac{2}{6} = \square$

$\frac{5}{5} - \frac{4}{5} = \square$

$\frac{5}{5} - \frac{2}{5} - \frac{1}{5} = \square$

$\frac{1}{5} \quad \frac{2}{5} \quad \frac{2}{4} \quad \frac{6}{10} \quad \frac{4}{6} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{8}{10} \quad \frac{5}{6} \quad 1$

**KUNSKAPSKRAV**  
**Begrepp** – har grundläggande kunskaper om begreppen liknämning och olika nämnare  
**Metod** – väljer fungerande matematisk metod för addition och subtraktion av tal i bråkform  
**Kommunikation** – använder bild som matematisk uttrycksform

62 Taluppfattning och tals användning – rationella tal och deras egenskaper. Tal i bråkform och deras användning i vardagliga situationer

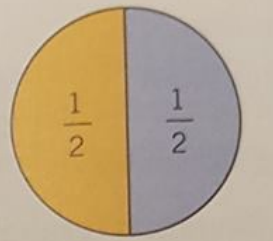
# Upper secondary grade 1 ("grade 10")

## 1.3 Tal i bråkform

### Hur stor andel?

Exempel Om du delar en pizza i två lika stora delar får du två halvor.

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$$



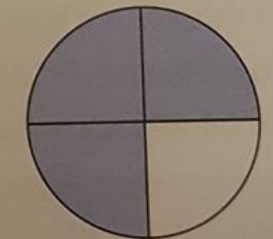
Om du delar pizzan i tre lika stora delar får du tre tredjedelar.

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$



Elna delar sin pizza i fjärdedelar och äter tre av delarna. Hur stor andel av pizzan äter hon?

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$



Hon äter tre fjärdedelar av pizzan.

Tre fjärdedelar är ett tal som i bråkform kan skrivas



Talet under bråkstrecket talar om vilka delar vi har (fjärdedelar).  
 Talet ovanför bråkstrecket talar om hur många delar vi har (3 stycken).

Omvandla tal i bråkform till decimalform kan vi enkelt göra med räknare. Tabellen visar några viktiga omvandlingar du bör kunna utantill!

	Bråkform	Decimalform
En halv	$\frac{1}{2}$	0,5
En tredjedel	$\frac{1}{3}$	0,333 ...
En fjärdedel	$\frac{1}{4}$	0,25
En femtedel	$\frac{1}{5}$	0,2

1+3  
0,3333333333

## 38. Division – delningsdivision



Charlie, Isa och Liam delar 12 godisbitar lika mellan sig.  
”En till Charlie, en till Isa, en till Liam, en till Charlie, en till Isa, en till Liam ...”

C I L C I L C I L C I L

Alla får fyra godisbitar var.

1. Dela godisbitarna lika mellan Charlie (C), Isa (I) och Liam (L). Hur många godisbitar får de var?

a.



Svar: 1 godisbit

b.



Svar: godisbitar

 Öva begreppen.

 Lyssna på berättelsen.

## 39. Division – innehållsdivision

 Film

 Lyssna på berättelsen.

Charlie lägger  
12 bullar i påsar.  
Han lägger 4  
bullar i varje påse.

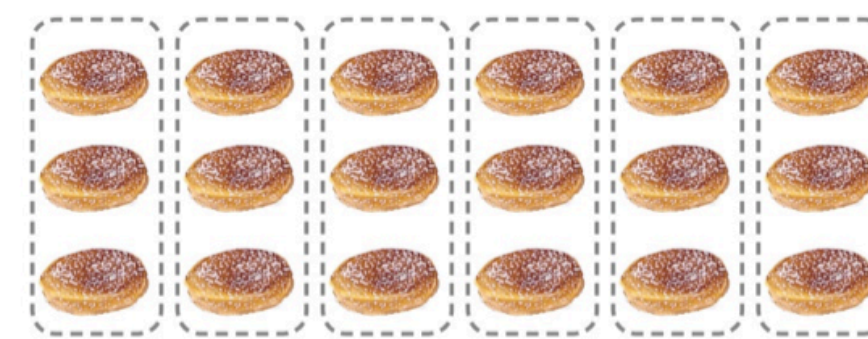
Hur många påsar  
behöver Charlie?

Svar: 3 påsar



1. Du lägger 18 bullar i påsar. Hur många påsar behöver du?

i. Det ska vara 3 bullar i varje påse.



Svar: påsar

b. Det ska vara 6 bullar i varje påse.



Svar: \_\_\_\_\_



## 40. Att skriva division

Lyssna på berättelsen.     
 Film

Det finns sex glasskolor.  
I varje bägare ska det vara  
2 klor glass. Hur många  
bägare behövs det?

Divisionen kan skrivas

$$\frac{6}{2} = 3 \quad \frac{\text{täljare}}{\text{nämnare}} = \text{kvot}$$

eller

$$6/2 = 3$$

täljare/nämnare = kvot

Du säger: Sex delat med två är  
lika med tre, eller sex dividerat  
med två är lika med tre.



1. Dividera.



a.  $\frac{8}{2} = \underline{\quad}$



b.  $\frac{9}{3} = \underline{\quad}$

## 41. Sambandet mellan division och multiplikation

Film  Lyssna på berättelsen.

Det finns 12 böcker. Hur  
många högar får du, om du  
lägger 4 böcker i varje hög?

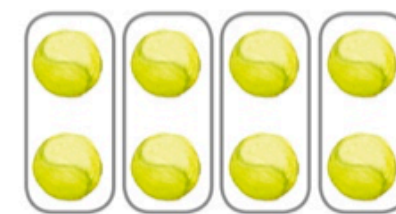
$$\frac{12}{4} = 3$$

Du kan kontrollera division  
med multiplikation.

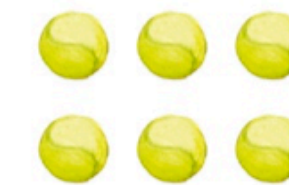
$$3 \cdot 4 = 12$$



1. Dividera. Kontrollera division med multiplikation.



a.  $\frac{8}{2} = \underline{4}$   
 $\underline{4} \cdot \underline{2} = \underline{8}$



b.  $\frac{6}{3} = \underline{\quad}$   
 $\underline{\quad} \cdot \underline{\quad} = \underline{6}$



c.  $\frac{12}{6} = \underline{\quad}$   
 $\underline{\quad} \cdot \underline{\quad} = \underline{\quad}$

Grade	Mathematical content	Concept	Procedure	Connection
3	division, equal sharing division, equal grouping division, equal grouping multiplication and division division, equal grouping proportionality fractions, geometric part whole fractions, geometric part whole fractions, geometric part whole fractions, one whole	$\text{☺}$ $\text{☺} \longrightarrow \text{♥}$ $\text{☺} \longrightarrow \pi$ $\text{☺} \longrightarrow \pi$ $\text{☺} \longrightarrow \pi$ $\text{☺} \longrightarrow \pi$ $\text{♥}$ $\text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$		$\text{♥} \longrightarrow \pi$
4	division, equal grouping and sharing division, short division division, short division fractions, equal fractios	$\text{☺} + \text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$	$\text{♥} \longrightarrow \pi$ $\longrightarrow \pi$	
5	fractions, mixed fractions fractions, reducing fractions fractions, reducing to lowest term dividing fraction with whole number multiply fraction with whole number fractions, fraction of numbers	$\text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$	$\longrightarrow \pi$ $\longrightarrow \pi$ $\longrightarrow \pi$ $\longrightarrow \pi$	$\text{☺} + \text{♥} \longrightarrow \pi$ $\text{☺} + \text{♥} \longrightarrow \pi$
6	fractions, geo. part whole, mixed fractions, convert to and from mixed fractions, reducing, reducing to lowest term fractions, expanding fractions	$\text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$ $\text{♥} \longrightarrow \pi$	$\longrightarrow \pi$ $\longrightarrow \pi$ $\longrightarrow \pi$	

Grade	Mathematical content	Concept	Procedure	Connection
7	fractions, geometric part whole	♥ → π		
	fractions, geometric part hole bigger than one	♥ → π		
	fractions, size of fractions	♥ → π	♥ → π	
	fractions, equal fractions	♥ → π		
	fractions, reduce fractions	♥ → π	→ π	
	fractions, expanding fractions	♥ → π	→ π	
	fractions, fraction of number	☺+♥ → π		
	fractions, geometric part whole	♥ → π		
	fractions, fractions bigger than one	☺+♥ → π		
	fractions, equal fractions	π, ♥ → π		
8	fractions, part whole of numbers	☺+♥ → π		
	fractions, equal fractions	♥ → π	→ π	
	multiply a fraction with a whole number		π	☺+♥ → π
	multiply fractions	♥ → π	→ π	
	multiply fractions		→ π	
	fractions	♥ → π		
	fractions, expand and reduce	♥ → π	→ π	
	fractions, reduce fractions with variables*	π	→ π	
	fractions, division of fractions*			☺+♥ → π
	fractions, division of fractions, inverse*	π, π → ♥	π	π, ♥ → π
fractions, division of fractions with variables*	π	π		
9	fractions, comparing fractions	♥ → π		
	fractions, equal fractions, reducing, expanding	♥ → π	→ π	
	fractions, multiply fractions		→ π	♥ → π

\* From the "red course" with advanced material that not all student will necessarily encounter.

# Polysemy

When one word has several but related meanings

When one concept has several but related meanings

Part-whole  
Division (partitive, equal sharing)  
Measuring  
Linearity  
Rate  
Ratio  
Proportionality

$$\frac{a}{b}$$

# On Proof and Progress in Mathematics

WILLIAM P. THURSTON

People have very different ways of understanding particular pieces of mathematics. To illustrate this, it is best to take an example that practicing mathematicians understand in multiple ways, but that we see our students struggling with. The derivative of a function fits well. The derivative can be thought of as:

- (1) Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.
- (2) Symbolic: the derivative of  $x^n$  is  $nx^{n-1}$ , the derivative of  $\sin(x)$  is  $\cos(x)$ , the derivative of  $f \circ g$  is  $f' \circ g * g'$ , etc.
- (3) Logical:  $f'(x) = d$  if and only if for every  $\varepsilon$  there is a  $\delta$  such that when  $0 < |\Delta x| < \delta$ ,

$$\left| \frac{f(x + \Delta x) - f(x)}{\Delta x} - d \right| < \delta.$$

- (4) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.
- (5) Rate: the instantaneous speed of  $f(t)$ , when  $t$  is time.
- (6) Approximation: The derivative of a function is the best linear approximation to the function near a point.
- (7) Microscopic: The derivative of a function is the limit of what you get by looking at in under a microscope of higher and higher power.

The list continues; there is no reason for it ever to stop.

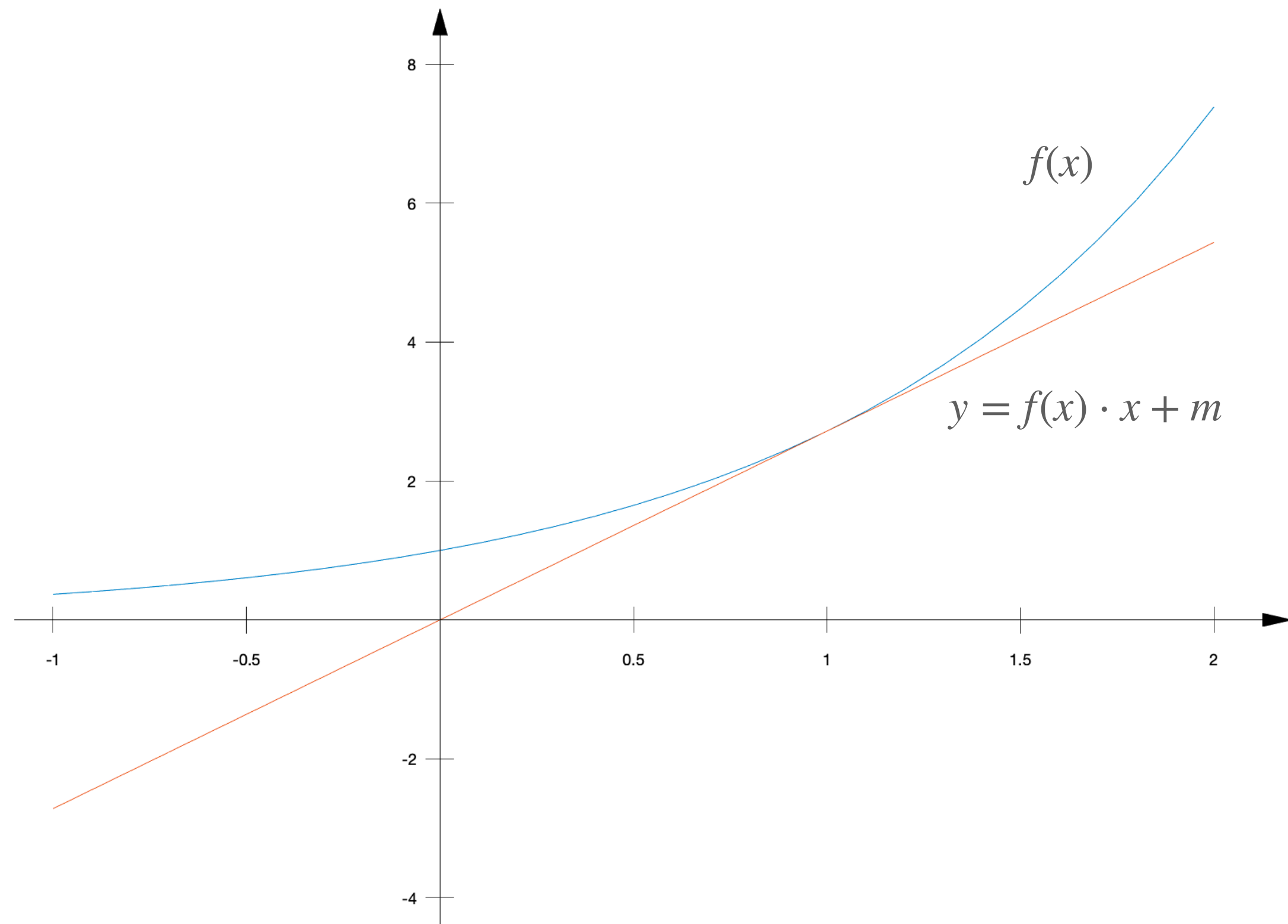
37. The derivative of a real-valued function  $f$  in a domain  $D$  is the Lagrangian section of the cotangent bundle  $T^*(D)$  that gives the connection form for the unique flat connection on the trivial  $\mathbf{R}$ -bundle  $D \times \mathbf{R}$  for which the graph of  $f$  is parallel.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

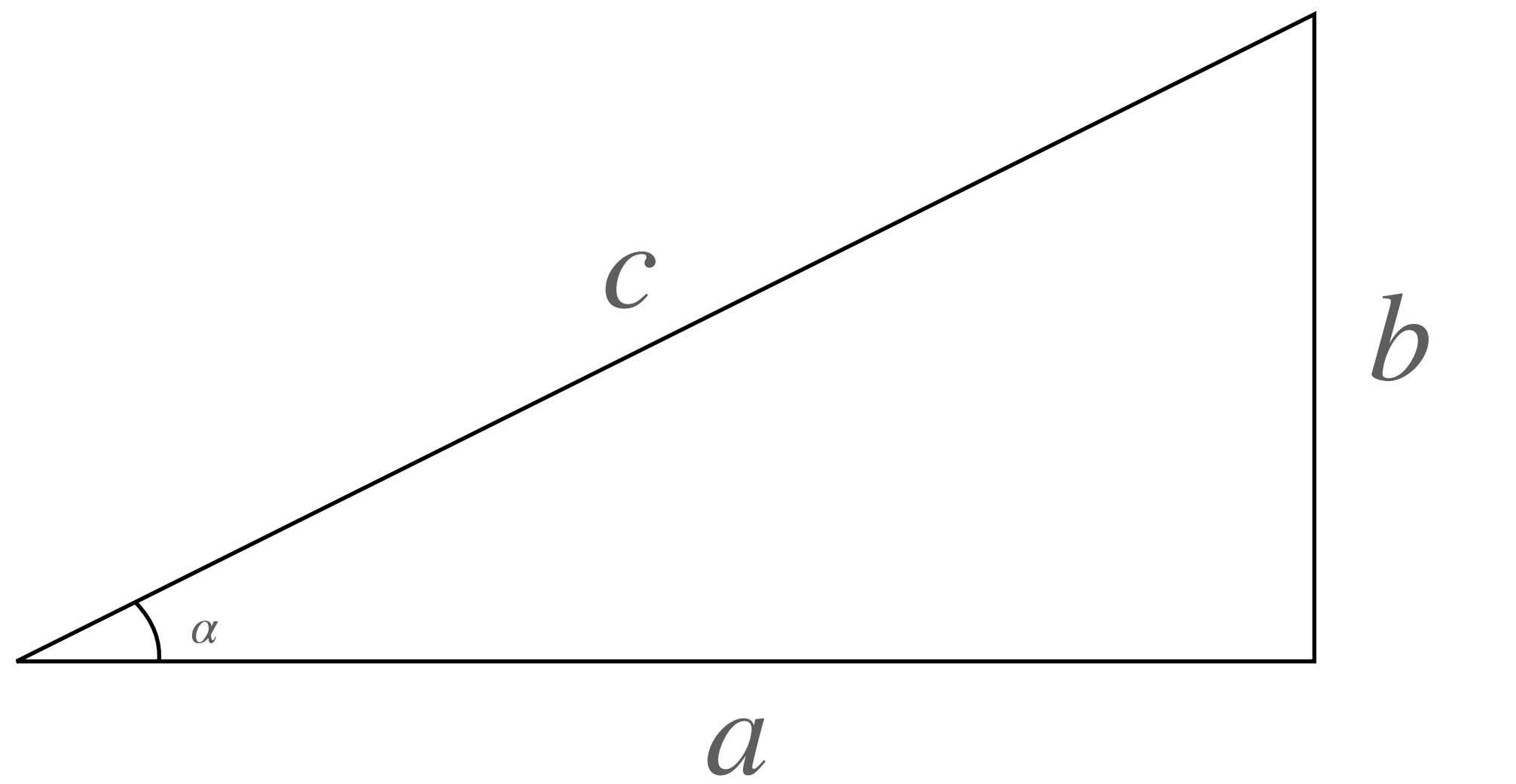
$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n, \quad e^x = ?$$

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

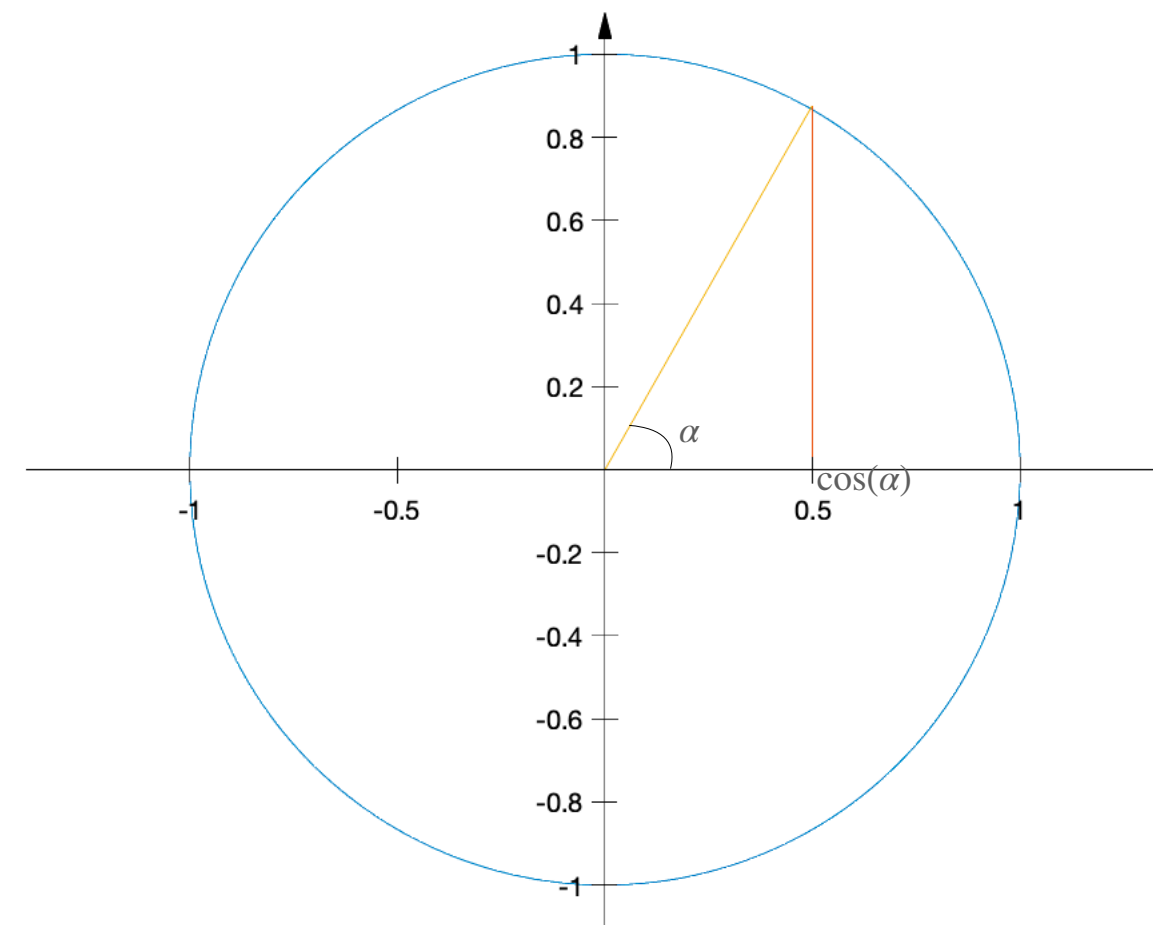
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$



$$e^x = y : y' = y, \quad y(0) = 1$$



$$\cos(\alpha) = a/c$$



$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$



# Consequences for teaching?

**Thank you for listening**